PROGRESS AT THE INTERFACE BETWEEN EFFECTIVE ONE BODY THEORY AND THE SMALL MASS RATIO APPROXIMATION

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**MOTIVATIONS**

- Various approximations to the two body problem [Post-Newtonian (PN), Post-Minkowskian (PM) and Small Mass Ratio (SMR)] have different domains of validity in the “compactness - mass ratio” parameter space.

- The Effective One Body (EOB) theory can extend these domains of validity.

- SMR terms linear in the symmetric mass ratio $\nu = \frac{q}{(q+1)^2}$, but at very high PN orders have been included in the EOB Hamiltonians. [Bini, Damour, Geralico, Kavanagh, …]

- We want a Hamiltonian which is not PN truncated and that contains information at linear order in $\nu$. [Akçay, Barausse, Buonanno, Damour, Le Tiec, van de Meent,…]
CURRENT EOB HAMILTONIAN FOR NON-SPINNING BLACK HOLES

• The EOB theory is based on an energy map linking the real two-body problem to an effective one:

\[ H_{EOB} = M \sqrt{1 + 2\nu \left( \frac{H_{\text{Eff}}}{\mu} - 1 \right)} \]

\[ \frac{H_{\text{Eff}}}{\mu} = \sqrt{A(u,\nu)[1 + p_\phi^2 u^2 + A(u,\nu)D(u,\nu)^{-1} p_r^2 + Q(u, p_r, \nu)]} \]

\[ M = M_1 + M_2 \quad \text{Total mass} \quad p_\phi \quad \text{reduced angular momentum} \quad u = 1 / R \quad \text{reduced inverse radius} \]

\[ \mu = M \nu \quad \text{Reduced mass} \quad p_r \quad \text{reduced radial momentum} \]

• At 2PN order, the effective body moves on a geodesic of a deformed Schwarzschild spacetime [Buonanno-Damour (1998)]. At 3PN order, non-geodesic terms must be inserted in a quartic-momenta term Q. [Damour-Jaranowski-Schaefer (2000)]

• Q depends in principle on \( p_r \) and \( p_\phi \). In DJS2000, Q only depends on \( p_r \). This is the DJS gauge.
THE LIGHT RING DIVERGENCE

• [Le Tiec et al.(2011), Barausse et al. (2011)] used the first law of binary black hole mechanics to calculate the linear in \( \nu \) correction to the potential \( A(u, \nu) \):

\[
A(u, \nu) = A_{\text{Schw}} + \nu a_{\text{SMR}}[u, z_{\text{SMR}}(u)] = 1 - 2u + \nu \left[ z_{\text{SMR}}(u)\sqrt{1 - 3u} - u \left( 1 + \frac{1 - 4u}{\sqrt{1 - 3u}} \right) \right]
\]

• The Detweiler redshift \( z_{\text{SMR}}(u) \) incorporates the SMR data in the EOB theory.

• Possible presence of a divergence in \( A(u, \nu) \) at the Schwarzschild LR \( (u_{\text{LR}} = 1/3) \). [Barausse et al. (2011)]

  Why do we expect the procedure to lead to a divergence?

• In the circular orbit limit, \( p_\phi \) is:

\[
\left. \frac{\partial H_{\text{EOB}}}{\partial u} \right|_{p_r=0} = 0 \Rightarrow \left. p_\phi^2 \right|_{\text{circ}} \sim (1 - 3u)^{-1}
\]

• At the LR, the circular orbit binding energy \( E_B\left|_{\text{circ}} = \frac{H_{\text{EOB}}\left|_{\text{circ}}}{\nu} - 1 \right. \) is dominated by:

\[
E_B\left|_{\text{circ}} \sim H_{\text{Eff}}^2\left|_{p_r=0} \frac{u \to u_{\text{LR}}}{a_{\text{SMR}}[u, z_{\text{SMR}}(u)] \times p_\phi^2\left|_{\text{circ}} \sim (1 - 3u)^{-3/2} \right. \to a_{\text{SMR}}[u, z_{\text{SMR}}(u)] \sim (1 - 3u)^{-1/2}}{\nu}
\]
THE LIGHT RING DIVERGENCE

• [Le Tiec et al.(2011), Barausse et al. (2011)] used the first law of binary black hole mechanics to calculate the linear in $\nu$ correction to the potential $A(u, \nu)$:

$$A(u,\nu) = A_{Schw} + \nu a_{SMR}[u, z_{SMR}(u)] = 1 - 2u + \nu \left[ z_{SMR}(u) \sqrt{1 - 3u} - u \left( 1 + \frac{1 - 4u}{\sqrt{1 - 3u}} \right) \right]$$

• The Detweiler redshift $z_{SMR}(u)$ incorporates the SMR data in the EOB theory.

• Possible presence of a divergence in $A(u, \nu)$ at the Schwarzschild LR ($u_{LR} = 1/3$). [Barausse et al. (2011)]

• [Akcay et al. (2012)] confirmed this divergence when data for $z_{SMR}(u)$ were made available up to the LR.
THE LIGHT RING DIVERGENCE

Why is the divergence a problem?

- The EOBSMR [DJS gauge] contains a divergence at the LR.
- Comparison with a Numerical Relativity (NR) simulation from [Ossokine et al. (2017)]
A NEW GAUGE

• [Damour (2017)] introduced the Energy gauge in the context of PM calculations.

\[ H_{\text{Eff}} = \sqrt{H^2_{\text{Schw}}} + \delta H^2_{\text{Eff}}[u,H_{\text{Schw}}(u,p_r,p_\phi)] \]

• The gauge depends on a new variable, the Schwarzschild Hamiltonian \( H_{\text{Schw}} \). In the circular orbit limit, \( H_{\text{Schw}} \) diverges at the LR, but it is regular at the LR for generic orbits.

\[ H_{\text{Schw}} = \sqrt{(1-2u)[1+p_\phi^2u^2+(1-2u)p_r^2]} \]

\[ p_\phi_{\text{circ}} = \frac{u(1-3u)}{\sqrt{1-3u}} \]

\[ p_r = 0 \]

\[ H_{\text{Schw}} \bigg|_{\text{circ}} = \frac{1-2u}{\sqrt{1-3u}} \]

• The key idea is to push the divergence onto \( H_{\text{Schw}} \), so to recover it only in the circular orbit limit, where we physically expect it.
1) Calculate linear in $V$, circular orbit binding energy as a function of frequency.

2) Equate to binding energy from [Le Tiec et al. (2011)] at fixed frequency.

3) Impose that the $X_i$ coefficients are smooth at the LR, in order to get:

\[
X_0 = \frac{z_0(u) - (1-4u)u}{(1-2u)^3}
\]

\[
X_1 = \frac{z_1(u) - u}{(1-2u)^2}
\]

\[
X_2 = \frac{z_2(u)}{(1-2u)^3}
\]

The $X_i$ coefficients are regular at the LR.

\[
\Rightarrow \text{The divergence has been absorbed by the Schwarzschild Hamiltonians.}
\]
We evolve EOB Hamiltonians via the Hamilton equations (with EOB flux $F_\phi$):

1) \[
\frac{dR}{dt} = \frac{A(R)}{\sqrt{D(R)}} \frac{\partial H_{EOB}}{\partial p_{R^*}}
\]

2) \[
\Omega = \frac{d\phi}{dt} = \frac{\partial H_{EOB}}{\partial p_\phi}
\]

3) \[
\frac{dp_{R^*}}{dt} = - \frac{A(R)}{\sqrt{D(R)}} \frac{\partial H_{EOB}}{\partial R} + F_\phi \frac{p_{R^*}}{p_\phi}
\]

4) \[
\frac{dp_\phi}{dt} = F_\phi
\]

Here the radius $R = 1/u$ is used. The radial momentum is calculated in tortoise coordinates.

We use $R(t)$ as a proxy for the behaviour of the EOBSMR dynamics at the LR.

The EOBSMR [DJS gauge] Hamiltonian is the one analytically calculated in [Barausse et al. (2012)].
We compare the fractional difference of energy $\frac{\Delta E_{\text{bind}}}{E_{\text{NR}}}$ (%) until merger between the EOB and NR as a function of the angular momentum.

- We stop the evolution at the Schwarzschild LR.
- NR data for the binding energy from [Ossokine et al. (2017)].
We compare the fractional difference of energy $\Delta E_{\text{bind}} / E_{\text{NR}}$ (%) until merger between the EOB and NR as a function of the angular momentum.

We stop the evolution at the Schwarzschild LR.

NR data for the binding energy from [Ossokine et al. (2017)].
CONCLUSIONS

WHAT WAS DONE:

- We built a first example of EOB Hamiltonian informed by the SMR approximation that contains terms linear in $\nu$. The Hamiltonian can be evolved smoothly through the LR.
- We found that the EOBSMR differs by around 2% from NR at merger.
- We found that EOB binding energy performs slightly better against NR when the DJS gauge, instead of the Energy gauge, is used.

TO DO:

- Better fit for the redshift, with new SMR data from M. van de Meent.
- Include higher orders in $\nu$ from PN expansion in the Energy gauge.
- Compare the Hamiltonian to a larger set of NR data to assess its accuracy.
- Build an EOBNR-SMR waveform model based on the new gauge.