The mathematics of multiscale expansions and the trajectory towards sub-radian accurate waveforms

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Main problem for getting post-adiabatic waveforms is bringing down the phase error

\[ \phi = \frac{1}{\epsilon} \left[ \phi^{(0)} + \sqrt{\epsilon} \phi^{(1/2)} + \epsilon \phi^{(1)} + \ldots \right] \]

resonances to be covered in future work

\( \phi^{(0)} \): adiabatic order is comparatively easy, just need dissipative first order self-force

\( \phi^{(1)} \): post-adiabatic order is delicate; scaling arguments suggest that we will require:
- Dissipative part of second-order self force
- Conservative first-order self force effects
- Slow deviation from geodesic motion (beyond osculating geodesics - maybe small see Peter Diener’s talk)
- Slow accretion of central mass and spin (\( \mathcal{O}(\mu) \) over full inspiral)

An osculating geodesics approach would neglect the true history and the evolution of spacetime (Note: not yet formulated at second order)
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Self-consistent includes full slow evolution of worldline, but (without modification) does not accurately track slow evolution of the spacetime
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Currently, multiscale method is the best suggested technique to directly compute all required effects
Prior work applying multiscale to EMRIs

- Mino and Price (2008)
  - A proof-of-concept computation of flat space Klein-Gordon scalar radiation-reaction for a quasicircular case, through post-adiabatic order

- Hinderer and Flanagan (2008)
  - First major step in computing the multiscale dynamics; presented the equations for the orbit itself assuming field solution input

- Pound (2015)
  - Returned to quasicircular scalar case, now with a quadratic source chosen to emulate gravitational nonlinearity
  - Uncovered and resolved a critical problem at large scales: an infrared divergence arises from periodicity construction
Zones and scales of approximation methods

- Mathematical preliminaries
- Multiscale methods for EMRIs
- **Near small object**: \( \bar{r} \ll M \)
Puncture [Pound, Miller], multiscale worldline
- **Interaction zone**: \( |r_*| \ll M/\epsilon \)
Multiscale wave equation
- **Far zone**: \( r_* \gg M \)
Geometric optics, with some Post-Minkowski techniques; Extending [Pound 2015]
- **Near-Horizon**: \( r_* \ll M \)
Black hole perturbation theory [Isoyama, Pound, Tanaka, Yamada]
- (future work) Resonances
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EMRIs as a weakly nonlinear oscillator

- A two-companion system with $\mu/M \equiv \epsilon \ll 1$

- The general problem:
  - An (instantaneously) nearly geodesic worldline, source scale $\sim \mu$
  - Generates weak metric perturbations $\sim \epsilon$
  - Weak metric perturbations satisfy a weakly nonlinear wave equation, from suppression of quadratic sources

- The motion is non-conservative
  - Gravitational radiation carries energy, angular momentum, and Carter to $I^+, H^+$

- Large separation of timescales $T_{\text{orbit}} \ll T_{\text{RR}}$
The general problem we wish to solve:

\[ D[f(t)] + \epsilon Q[f(t)] = 0, \]

with well-understood oscillator differential operator \( D \) and nonlinear operator \( Q \).

A naive approximation takes,

\[ f(t) \approx f^{(0)}(t) + \epsilon f^{(1)}(t) + \ldots, \]

where

\[ D[f^{(0)}(t)] = 0 \]

However, typically \( D \) is conservative, while \( \epsilon Q \) introduces dissipation, so at late times,

\[ \epsilon f^{(1)} \sim f^{(0)} \]
Dissipation from weak nonlinearity is slow compared to oscillations, 
\[ T_{\text{diss}} \sim \epsilon T_{\text{osc}} \]

General procedure:
- Introduce a pair of time variables \( \{ \varphi, \tilde{t} \} \), strictly periodic \( \varphi, \tilde{t} = \epsilon t \)
- Promote all physical variables \( f(t) \rightarrow f'(\varphi, \tilde{t}) \)
- Promote all differential operators \( D \rightarrow D' = D|_{\partial_t \rightarrow \Omega(\tilde{t}) \partial_\varphi + \epsilon \partial_{\tilde{t}}} \)
- Solve differential equations order-by-order for \( f^{(n)}(\varphi, \tilde{t}) \) and \( \Omega^{(n)}(\tilde{t}) \)

Projection to physical solution \( f(t) \) via,
\[
\tilde{t} = \epsilon t \quad \frac{d}{dt} \varphi = \Omega(\tilde{t})
\]
Physical time extracts full decay, depends on both slow and fast times.

Fast-time slice at $\tilde{t} = 3$

Fast-time slice at $\tilde{t} = 2$

Fast-time slice at $\tilde{t} = 1$

Fixed fast time, varying slow time

Fixed slow time, varying fast time

Values in full twotime space

Post-adiabatic multiscale Cornell University
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Multiscale approximation construction

- Multiscale approximation promotes time dependence to multiple (3 fast, 1 slow) variables $t \rightarrow \{\tilde{w}, q^A\}$,

$$w = t + \alpha(r)$$

$$\tilde{w} = \frac{\mu}{M} w \equiv \epsilon w$$

$$\frac{d\varphi^A}{dw} = \Omega^A(\tilde{w}, \epsilon)$$

- Action angle variables $\varphi^A = q^A + \mathcal{O}(\epsilon^2)$ used from celestial mechanics solutions

- Metric and worldline ansatz:

$$g_{\alpha\beta} = g_{\alpha\beta}^{(0)}(x^i) + \epsilon h_{\alpha\beta}^{(1)}(\tilde{w}, q^A, x^i) + \epsilon^2 h_{\alpha\beta}^{(2)}(\tilde{w}, q^A, x^i) + \mathcal{O}(\epsilon^3)$$

$$z^\mu = z^{(0)}(\tilde{w}, q^A) + \epsilon z^{(1)}(\tilde{w}, q^A) + \mathcal{O}(\epsilon^2)$$

- Incorporating slow-time evolution ($\tilde{w}$) into the leading solutions preserves quality of approximation for the entire inspiral $\sim M^2/\mu$, and incorporates radiation-reaction worldline into subleading source
Problems at long distances: rapid slow time transmission

- For each slow time, we find the appropriate fast-time solution
  \[ h^{(n)}(x^i, \tilde{w}, q^A) \]

- True evolution assembled from a path through the 4-dimensional \( \{\tilde{w}, q^A\} \)

- Quasi-conserved quantities \( \{E^{(0)}(\tilde{w}), L_z^{(0)}(\tilde{w}), Q^{(0)}(\tilde{w})\} \) are constant over a surface of constant \( \tilde{w} \)
  - For spacelike constant \( \tilde{w} \) surfaces, transmission of information to distances of \( \sim M/\epsilon \) in times of \( \sim M \) unphysical

Steady frequency on spacelike surfaces of constant \( \tilde{t} \) unphysical
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Solution: enforce surfaces of constant \( \tilde{w} \) are asymptotically null:

- Transmission to near-retarded time at \( I^+ \) and near-advanced time at \( H^+ \) acceptable, approximation convergence restored.
Breakdown at long distances: extended source

- At large scales of integration domain, another more subtle problem causes a failure of convergence [Pound 2015]

- Multiscale assumes radiation timescale longer than all other time scales

- At each order we solve a wave equation of the form

\[ \Box q^A h_{\mu\nu} = S(x^i, q^A, \tilde{w}), \]

for some source \( S \).

- At long scales, inverting \( \Box q^A \) assumes an eternal source (in \( q^A \)), so fills space with radiation

- Leading second-order source scales as \( \sim \Omega^2 / r^2 \)
  - Leads to a divergent second order solution if taken over full spatial domain
  - Divergence arises even with asymptotically null time variable [Pound 2015]

- A separate approximation is needed for \( |r^*| \gg M \)
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- Perform an action-angle variable decomposition for each fixed \( \tilde{w} \), including source terms determined by slow time derivatives

- Forcing terms are determined form self-acceleration as

\[
\begin{align*}
g^A &= \frac{\partial q^A}{\partial p^\mu} a^\mu \\
F^M_{\mu} &= \frac{\partial J^M}{\partial P^N} \frac{\partial P^N}{\partial p^\mu} a^\mu,
\end{align*}
\]

- Finally, action-angle variables obey the multiscale equations:

\[
\begin{align*}
\frac{dq^A}{dw} &= \Omega^A = \omega^A [P^{(0)M}(\tilde{w}) + \epsilon P^{(1)M}(\tilde{w}, q^A) + \ldots] \\
&\quad + \epsilon g^{(1)A}(q^A, P^M) + \epsilon^2 g^{(2)A}(q^A, P^M) + O(\epsilon^3) \\
\frac{dJ^M}{dw} &= \epsilon G^{(1)M}(q^A, P^M) + \epsilon^2 G^{(2)M}(q^A, P^M) + O(\epsilon^3)
\end{align*}
\]
Near-identity transformations

▶ The idea: perform a small alteration to the action-angle \( \{q^A, J^M\} \) to simplify the equations of motion

▶ Recently shown to have significant practical importance for rapid computations [Van de Meent, Warburton 2018] See Niels’ talk next

▶ Can be used to entirely eliminate [Flanagan, Vines] the angle-variable dependence of self force terms,

\[
J'^M = J^M + \epsilon \frac{i \tilde{G}^M_{kA}}{k^A \omega_A} \\
q'^A = q^A + \epsilon \frac{i}{k^A \omega_A} \left( \tilde{g}^A_{kA} - \frac{\partial \omega^A}{\partial J^M} T^M_{kA} \right)
\]

▶ Resulting equations of motion have only zero-frequency forcing terms,

\[
\frac{\partial q'^A}{\partial w} = \omega^A (P^M) + \epsilon g^{(1)A} (J'^M) + \epsilon^2 g^{(2)A} (J'^M) \\
\frac{\partial J'^M}{\partial w} = \epsilon G^{(1)} (J'^M) + \epsilon^2 G^{(2)} (J'^M)
\]
Puncture correction from subleading worldline

- Require the puncture metric $h^{P}_{\alpha\beta}$ through second order
- Formulated in [Pound, Miller] in terms of distance to exact worldline $h^{P}(z)$
- Instead, for two timescale, worldline is perturbatively expanded
  
  $z^{\mu} = z^{(0)\mu}(q^{A}, \tilde{w}) + \epsilon z^{(1)\mu}(q^{A}, \tilde{w}) + \mathcal{O}(\epsilon^{2})$

- Gives an $\mathcal{O}(\mu)$ displacement from fiducial worldline $\Rightarrow$ dipole correction
- Expansion of covariant puncture accomplished via techniques presented in [Pound 2015], adjusted to coordinate multiscale time derivatives

$\Delta Z$ of center of mass $\Rightarrow$ dipole
Mapping back to worldline expansion

- Corrections to puncture require an explicit form of \( z^{(1)}(q'^A, J'^M) \) not explicitly given in action-angle equations of motion

- To obtain this inversion, we perturbatively expand

\[
\frac{dz^i}{dw} = \frac{p_\beta g^{i\beta}}{p_\beta g^{w\beta}}
\]

- Requires information from leading and subleading frequencies \( \Omega^{(0)}, \Omega^{(1)} \)
  - Subleading frequencies include self-force contributions \( \langle g^A \rangle \)
  - Action variable frequencies \( \partial H / \partial J^A \equiv \omega^A \) determined by inverting

\[
\frac{\partial J^\alpha}{\partial P^\gamma} \frac{\partial P^\gamma}{\partial J_\alpha} = \delta^{\gamma}_{\beta}
\]

- Oscillatory dependence of self forces \( g^A \) and \( G^M \) must be restored in order to obtain full fast-time orbits - all \( p^{(1)}, \Omega^{(1)} \) depend explicitly on both \( J'^M, q'^A, g^A, G^M \) directly
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Two timescale wave equations for Lorenz gauge

- Practical computations using explicit EFE will likely be performed in Lorenz gauge, promoted to multiscale

\[ \nabla^{(0)}_\mu h^{(1)\mu\nu} = 0 \]
\[ \nabla^{(1)}_\mu h^{(1)\mu\nu} + \nabla^{(0)}_\mu h^{(2)\mu\nu} = 0 \]

- Imposition of Lorenz gauge gives multiscale relaxed EFE expansion

\[ \delta E^{(0)}_{\mu\nu} [h^R^{(1)}] = - \delta E^{(0)}_{\mu\nu} [h^P^{(1)}] + 8\pi \bar{T}_{\mu\nu} \equiv S^{R_{\text{eff}}^{(1)}}_{\mu\nu} \]
\[ \delta E^{(0)}_{\mu\nu} [h^R^{(2)}] = - \delta E^{(0)}_{\mu\nu} [h^P^{(2)}] - \delta^2 E^{(0)}_{\mu\nu} [h^{(1)}, h^{(1)}] - \delta E^{(1)}_{\mu\nu} [h^{(1)}] \equiv S^{R_{\text{eff}}^{(2)}}_{\mu\nu} \]

- Effective source formalism - recall talks by Peter Diener, Seth Hopper
- Puncture metric determined by $z^\mu$ expansion, discussed earlier
- Corrections to geodesic motion incorporated via $E^{(1)}$ terms
Teukolsky-Lousto-Campanelli overview

- Teukolsky-Lousto-Campanelli formalism offers a way of computing Weyl scalars $\psi_{0/4}^{(1)}$, $\psi_{0/4}^{(2)}$ without first finding the respective metric perturbations.

- first order:
  
  \[
  W_{+2}^{(0)} [\psi_{0}^{(1)}] = 4\pi \Sigma T_{+2} \\
  W_{-2}^{(0)} [\rho^{-4} \psi_{4}^{(1)}] = 4\pi \Sigma T_{-2}
  \]

- second order:
  
  \[
  W_{+2}^{(0)} [\psi_{0}^{(2)}] = S_{+2} [h^{(1)}] - W_{+2}^{(1)} [\psi_{0}^{(1)}] \\
  W_{-2}^{(0)} [\rho^{-4} \psi_{4}^{(2)}] = S_{-2} [h^{(1)}] - W_{-2}^{(1)} [\rho^{-4} \psi_{4}^{(1)}]
  \]

- Second-order source depends on all components of $h^{(1)}$ - must be reconstructed

- TLC equations do not explicitly restrict $\ell < 2$, static completion must be inferred [Merlin et. al.]
  
  - Slow variation can be computed from reconstructed $h^{(1)}$
A sketch of the implementation details, not yet thoroughly developed:

- Need to reconstruct $h^{(1)}$, so prefer leading TLC to be on the physical pointlike source, rather than effective source
  - Sharp feature at all $\ell$, at radius of source, require EHS [Barack, Ori, Sago] and transition condition from source recall from talks by Maarten, Zachary
  - Static completion part obtained by [Merlin et. al.]

- Expect second-order equations to become ill-defined without regularization, so effective source must be used

- At second order, an extended inhomogeneous source as well as a sharp feature at the radius of the orbit
  - Use extended particular solutions [Hopper, Evans]: separately get regular solution by integrating separation of vars from outside and from inside, transition at sharp feature
Slow variations for spacetime

- General scaling: $O(\epsilon^2)$ flux, $\mathcal{M}/\epsilon$ time
  - $O(\epsilon \mathcal{M})$ alteration in spacetime moments over time

- leading order must include $\delta \mathcal{M}, \delta a$
  \[
  h^{(1)}_{\alpha\beta} = \delta \mathcal{M}(\tilde{w}) \frac{\partial g_{\alpha\beta}}{\partial \mathcal{M}} + \delta(\mathcal{M}a)(\tilde{w}) \frac{\partial g_{\alpha\beta}}{\partial (\mathcal{M}a)} + \mathcal{F}_{\alpha\beta}(\mathcal{P}^M, q^A)
  \]

- What about other secular parts, like spin orientation, overall boost, or more subtle ‘charges’ from BMS:
  - Each of these introduce a slow-time dependent $\delta h^{(1)}_{\alpha\beta}(\tilde{w})$, but each can (at fixed $\tilde{w}$) be removed with a gauge transformation

  $\Rightarrow$ up to gauge, all of these effects give rise to a non-removable $\delta h^{(2)}_{\alpha\beta}(\tilde{w})$, but that’s post-2-adiabatic.
Determining $\delta M$, $\delta a$ in Lorenz gauge

- Consider the additional quasistatic part of the metric perturbation separately, permit a different gauge:

$$h^{(1)} = \frac{\partial g^{(0)}}{\partial M} \delta M(\tilde{w}) + \frac{\partial g^{(0)}}{\partial (aM)} \delta(aM)(\tilde{w}) + \mathcal{F}^{(1)}(x^i, P^M, q^A)$$

- $E_{\mu\nu}$ annihilates the $\partial g^{(0)}$ parts of variation

- In the multiscale Lorenz gauge, the gauge condition becomes dynamical

$$\nabla^{(1)}_{\mu} h R^{(1)\mu\nu} + \nabla^{(0)}_{\mu} h R^{(2)\mu\nu} = 0$$

- In a general sense, this condition is the constraint which preserves stress-energy conservation on the long scale of the inspiral
Instead of the Lorenz gauge giving conservation information, more generally we need to consult the EFE itself.

The quasistatic $\ell = 0$ part of the EFE determines the slowly varying parts
\[
\int d^3 q d^2 \Omega R_{tr}^{(1)} [h^{(1)}] = \alpha(r) \partial_{\tilde{u}} \delta M + \beta(r) \partial_{\tilde{u}} \delta a(\tilde{u})
\]
\[
\int d^3 q d^2 \Omega R_{\phi r}^{(1)} [h^{(1)}] = \gamma(r) \partial_{\tilde{u}} \delta M + \beta(r) \partial_{\tilde{u}} \delta a(\tilde{u})
\]

These can then be inverted with the EFE to obtain formulas in terms of the second-order Ricci $R[h^{(1)}, h^{(1)}]$.

Note that these derivations intuitively require metric reconstruction to determine quadratic ‘fluxes’.
Fluxes for orbital dynamics: overview

- First order version initially developed by [Gal'tsov; Sago, Fujita; Ganz et. al.]

- At first order, the balance law relations imply give an equality of conserved quantities at the orbit and asymptotic fluxes

\[
\left\langle \frac{d\mathcal{E}^{(0)}}{d\tilde{\tau}} \right\rangle = \left\langle u^\alpha u^\beta \mathcal{L}_\xi h^{(1)}_{\alpha\beta} \right\rangle
\]

\[
\Rightarrow \left\langle \frac{dE}{d\tilde{\tau}} \right\rangle = \sum i\omega \left( \alpha \left( Z^{(1)}_{\text{out}} \right)^2 + \beta \left( Z^{(1)}_{\text{down}} \right)^2 \right)
\]

\[
\left\langle \frac{dL_z}{d\tilde{\tau}} \right\rangle = \sum im \left( \alpha \left( Z^{(1)}_{\text{out}} \right)^2 + \beta \left( Z^{(1)}_{\text{down}} \right)^2 \right)
\]

- A similar identity holds for Carter constant evolution [Mino et. al.]

- At second order, we should anticipate a similar description, but with corrections

  - “Schott” terms from trapped energy in the system

"trapped" energy and angular momentum gives rise to Schott terms
Using the quadratic contribution to second-order self-force [Pound], we derive the second order form of flux balance:

\[
\left\langle \frac{d\mathcal{E}^{(1)}}{d\tilde{\tau}} \right\rangle = \frac{1}{2} \left\langle u^\alpha u^\beta \mathcal{L}_\xi h^{(2)}_{\alpha\beta} \right\rangle + \frac{1}{8} \left\langle u^\alpha u^\beta u^\gamma u^\delta \mathcal{L}_\xi \left( h^{(1)\mathcal{R}}_{\alpha\beta} h^{(1)\mathcal{R}}_{\gamma\delta} \right) \right\rangle \\
- \partial_{\tilde{\tau}} \left\langle \xi^\beta u^\gamma h^{(1)\mathcal{R}}_{\beta\gamma} \right\rangle - \frac{1}{2} \partial_{\tilde{\tau}} \left\langle \mathcal{E} u^\alpha u^\beta h^{(1)\mathcal{R}}_{\alpha\beta} \right\rangle
\]

This is gauge invariant as per full gauge transformation from [Pound '15].

We are currently developing a version for Carter constant as well.

Additional manipulation expresses this as a sum of contributions:

- Direct quadratic fluxes from \( h^{(2)\mathcal{R}} h^{(1)\mathcal{R}} \) products
- Corrections associated deviations from homogeneity of \( h^{(1)\mathcal{R}} \)
- Integrals over instantaneous in \( \tilde{w} \) worldline of
  - Quadratic terms in \( h^{(1)} \)
  - Terms with \( h^{(1)} \) multiplied by gauge vector to Rad. gauge \( \zeta \)
  - Terms \( \sim \partial_{\tilde{w}} h^{(1)} \)
  - Terms \( \sim \partial_{\tilde{w}} \zeta \)
Computational cost?

- A great deal of information can be ‘cached’ by analogy to the osculating geodesics formalism
  - First order solution is a rigorously correct interpolation across instantaneously geodesic solutions
  - The interpolation requires highly accurate frequencies $\Omega(\tilde{w})$, which require second-order solutions
- Second order is a combination of a part sourced also by instantaneously-geodesic contributions and a part which involves explicit time derivatives
- Parameter space: $\{\epsilon, a, \delta M, \delta a, E^{(0)}, E^{(1)}, L_z^{(0)}, L_z^{(1)}, Q^{(0)}, Q^{(1)}\}$
  - Perhaps this seems a bit daunting?
  - Note that the internal spacing in each dimension of leading and subleading parameters does not have to be as small as if we used $E = E^{(0)} + E^{(1)}$, for which we would need spacings $\ll \epsilon \mu$
  - Something I’d be interested in hearing discussion and objections from those that might consider implementations of multiscale
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Geometric optics for the far zone

- Spatial scales vary with $\tilde{x}^i \sim \epsilon x^i$, on scale with slow inspiral

- Construct ansatz with single fast variation parameterized by scalar function $\Theta(x^\nu) / \epsilon$

$$g_{\mu\nu}(x^\nu, \epsilon) = \epsilon^{-2} \left( \eta_{\mu\nu} + \epsilon h_{\mu\nu} [\tilde{x}^\nu] + \epsilon^2 j_{\mu\nu} \left[ \tilde{x}^\nu, \frac{\Theta}{\epsilon} \right] 
+ \epsilon^3 k_{\mu\nu} \left[ \tilde{x}^\nu, \frac{\Theta}{\epsilon} \right] + \mathcal{O}(\epsilon^4) \right)$$

- The rescaling of the coordinates grants an additional order to the weak waves, as they depend on $1/r = \epsilon / \tilde{r}$

- At leading order, the wave equation for this expansion gives simple $1/\tilde{r}$ radiation dependence

$$\frac{1}{\tilde{r}} \partial_{\tilde{r}} j_{AB} + \partial_{\tilde{r}} \partial_{\tilde{r}} j_{AB} = 0$$

- Subleading Lorenz gauge condition constrains additional components of $j$

- The geometric optics EFE at subleading order fixes the nonvanishing non-TT parts of oscillatory $k_{\mu\nu}$
Third order equations - quasistatic $j_0$

- Impose Lorenz gauge on the quasistatic part $j_0$

- Background correction + General wave equation

$$\square j_0^{\mu\nu}[\tilde{x}^\nu] + R_\mu^{\quad \nu^\rho} j_0^{\sigma\rho} = - \left\langle G^{(2,2)}_{\mu\nu}[j, j] \right\rangle$$

- Solvable via techniques first introduced by [Blanchet and Damour]

- Particular retarded solution written as integral:

$$j_0 = \text{FP}_{B \to 0} \left[ \frac{1}{K(B)} \int_\infty^{\tilde{r}} \tilde{z} \frac{S^{(k)}(\tilde{t} - \tilde{z})}{\tilde{r}^k} \frac{\partial L}{\partial \tilde{z}} \left( \frac{(\tilde{z} - \tilde{r})^{B-k+l+2}}{\tilde{t}} - \frac{(\tilde{z} + \tilde{r})^{B-k+l+2}}{\tilde{r}} \right) \right]$$

- With some manipulation, we can re-write the retarded solution as a further split of homogeneous + particular solution

$$j_{0,\ell} = \tilde{\partial}_L \frac{j_\ell^G(u)}{\tilde{r}} + j_\ell^H(u)$$

- Quasistatic $j$ match inward to the interaction zone to inform quasistatic mode boundary conditions
Evaluate integral assuming large $\tilde{r}$. Geometric optics construction gives $G^{(2,2)} \sim \tilde{r}^{-2}$

$$j^H_\ell = \frac{\hat{n}_L}{\tilde{r}} \int_0^\infty d\tilde{z} \left( \frac{1}{2} \ln \frac{\tilde{z}}{2\tilde{r}} + \sum_{n=1}^{\ell} \frac{1}{n} \right) \left\langle G^{(2,2)}[j, j] \right\rangle + \mathcal{O}(\tilde{r}^{-2} \ln(\tilde{r}))$$

$$\tilde{\partial}_L j^G_\ell(\tilde{u}) = \tilde{\partial}_L \frac{1}{\tilde{r}K_k} \int_{-\infty}^{\tilde{u}} d\tilde{s} \left\langle G^{(2,2)} \right\rangle (\tilde{s})(\tilde{u} - \tilde{s})^{\ell}$$

Scales similarly with $\varepsilon$ to outgoing waves - ‘memory’-like effect

Scaled coordinates $\tilde{x}$ explicitly incorporate the long scale dependence of the system

Region of nonlinear source $r \sim M/\varepsilon \Rightarrow \tilde{r} \sim M$
Zones and scales of approximation methods

- Mathematical preliminaries
- Multiscale methods for EMRIs
- Near small object: $\bar{r} \ll M$
  - Puncture [Pound, Miller], multiscale worldline
- Interaction zone: $|r_*| \ll M/\epsilon$
  - Multiscale wave equation
- Far zone: $r_* \gg M$
  - Geometric optics, with some Post-Minkowski techniques;
    Extending [Pound 2015]
- Near-Horizon: $r_* \ll -M$
  - Black hole perturbation theory
    [Isoyama, Pound, Tanaka, Yamada]
- (future work) Resonances
A resonant orbit is one in which two characteristic frequencies (e.g. $\Omega^r$ and $\Omega^\theta$) are related by a rational value.

In the multiscale formalism, resonances cause orbit-averages to develop $O(1)$ corrections.

Scaling arguments indicate that the duration of the resonant alteration should be $\sim M/\sqrt{\epsilon}$.

Over the course of the resonance, the orbit obtains a phase correction $\sim 1/\sqrt{\epsilon}$ and a geodesic parameter $P^M$ correction of $\sim \sqrt{\epsilon} \mu$ [Flanagan, Hinderer 2010].

Significant phase errors will result from ignoring a resonance should it be present.
Resonances: We really can’t avoid them

- Low-order resonances occur frequently in the geodesic parameter space, particularly dense near the ISCO [Brink, Geyer, Hinderer 2015]

- As shown by a study by [Ruangsri, Hughes 2014]
  - low-order resonances are ubiquitous in parameter space
  - The 3:1 resonance, very close to the ISCO occurred for all cases examined

- Order of resonance enters scale of effect as $\Delta \varphi \sim 1/\sqrt{(n + m)\epsilon}$
  - We may need to track resonances to order $|n| + |m| \sim \log(\epsilon)$ [From general scaling arguments from Arnold et. al.]
The ideas of what we can do

- Multiscale is not invalidated in the case of a resonance, just in need of correction

- Before and after the resonance, the standard non-resonant strategy holds, but needs $\sqrt{\epsilon}$ scale terms

  \[ P^M = P^{(0)M} + \sqrt{\epsilon}P^{(1/2)M} + \epsilon P^{(1)M} + O(\epsilon^{3/2}) \]

  \[ q^A = \frac{1}{\epsilon} \left( q^{(0)A} + \sqrt{\epsilon}q^{(1/2)A} + \epsilon q^{(1)A} + O(\epsilon^{3/2}) \right) \]

- We will also need the 'jumps' across the resonances in the phase and geodesic parameters [see computation by Van de Meent 2014]

- In general, during the transient resonance, a third time scale emerges $\hat{t} \sim \sqrt{\epsilon}t$, and the dynamics can be computed using a multiscale expansion
We now have a nearly complete, comprehensive framework for multiscale approximations. The description of the interaction zone and far zone are now well-understood. We are currently working steadily towards publication of a complete (hopefully implementation-friendly) description. Several methods work in concert to form a globally valid approximation scheme. Multiscale approximations are the only current technique which hold the promise to capture all post-adiabatic effects consistently. Future work for multiscale: resonances: generally introduce powers $\epsilon^{1/2}$