The mathematics of multiscale expansions and the trajectory towards sub-radian accurate waveforms

J. Moxon[1] E. Flanagan[1] T. Hinderer[3] A. Pound[2]

[1]Cornell University
Department of Physics
[2]University of Southampton
Mathematical Sciences
[3]University of Maryland, College Park
Maryland Center for Fundamental Physics

Capra 2018

Multiscale for EMRIs - why it's needed

- Main problem for getting post-adiabatic waveforms is bringing down the phase error
$$\phi = \frac{1}{\epsilon}[\phi^{(0)} + \sqrt{\epsilon}\phi^{(1/2)} + \epsilon\phi^{(1)} + \dots]$$
- resonances to be covered in future work
- $\phi^{(0)}$: adiabatic order is comparatively easy, just need dissipative first order self-force
- $\phi^{(1)}$: post-adiabatic order is delicate; scaling arguments suggest that we will require:
  - Dissipative part of second-order self force
  - Conservative first-order self force effects
  - Slow deviation from geodesic motion (beyond osculating geodesics - maybe small **see Peter Diener's talk**)
  - Slow accretion of central mass and spin ($\mathcal{O}(\mu)$ over full inspiral)
- An osculating geodesics approach would neglect the true history and the evolution of spacetime (Note: not yet formulated at second order)

Multiscale for EMRIs - why it's needed

- Main problem for getting post-adiabatic waveforms is bringing down the phase error
$$\phi = \frac{1}{\epsilon}[\phi^{(0)} + \sqrt{\epsilon}\phi^{(1/2)} + \epsilon\phi^{(1)} + \dots]$$
- resonances to be covered in future work
- $\phi^{(0)}$: adiabatic order is comparatively easy, just need dissipative first order self-force
- $\phi^{(1)}$: post-adiabatic order is delicate; scaling arguments suggest that we will require:
  - Dissipative part of second-order self force
  - Conservative first-order self force effects
  - Slow deviation from geodesic motion (beyond osculating geodesics - maybe small **see Peter Diener's talk**)
  - Slow accretion of central mass and spin ($\mathcal{O}(\mu)$ over full inspiral)
- Self-consistent includes full slow evolution of worldline, but (without modification) does not accurately track slow evolution of the spacetime

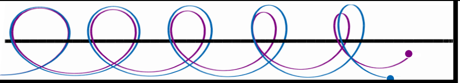
Multiscale for EMRIs - why it's needed

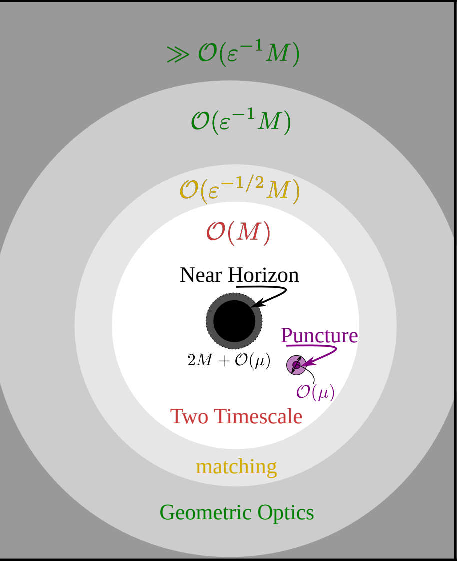
- Main problem for getting post-adiabatic waveforms is bringing down the phase error
$$\phi = \frac{1}{\epsilon}[\phi^{(0)} + \sqrt{\epsilon}\phi^{(1/2)} + \epsilon\phi^{(1)} + \dots]$$
- resonances to be covered in future work
- $\phi^{(0)}$: adiabatic order is comparatively easy, just need dissipative first order self-force
- $\phi^{(1)}$: post-adiabatic order is delicate; scaling arguments suggest that we will require:
  - Dissipative part of second-order self force
  - Conservative first-order self force effects
  - Slow deviation from geodesic motion (beyond osculating geodesics - maybe small **see Peter Diener's talk**)
  - Slow accretion of central mass and spin ($\mathcal{O}(\mu)$ over full inspiral)
- Currently, multiscale method is the best suggested technique to directly compute all required effects

Prior work applying multiscale to EMRIs

- Mino and Price (2008)
  - A proof-of-concept computation of flat space Klein-Gordon scalar radiation-reaction for a quasicircular case, through post-adiabatic order
- Hinderer and Flanagan (2008)
  - First major step in computing the multiscale dynamics; presented the equations for the orbit itself assuming field solution input
- Pound (2015)
  - Returned to quasicircular scalar case, now with a quadratic source chosen to emulate gravitational nonlinearity
  - Uncovered and resolved a critical problem at large scales: an infrared divergence arises from periodicity construction

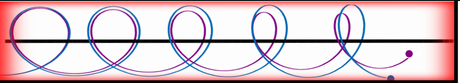
Zones and scales of approximation methods

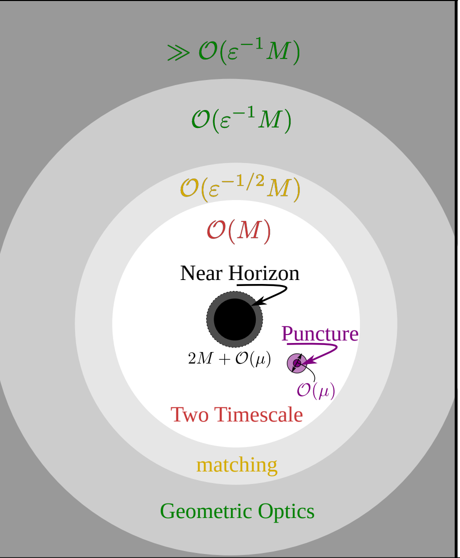




- Mathematical preliminaries
- Multiscale methods for EMRIs
- Near small object : $\bar{r} \ll M$
  Puncture [Pound,Miller], multiscale worldline
- Interaction zone: $|r_*| \ll M/\epsilon$
  Multiscale wave equation
- Far zone: $r_* \gg M$
  Geometric optics, with some Post-Minkowski techniques; Extending [Pound 2015]
- **Near-Horizon**: $r_* \ll -M$
  Black hole perturbation theory [Isoyama,Pound,Tanaka,Yamada]
- (future work) Resonances

Zones and scales of approximation methods

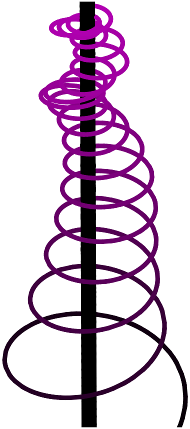




- Mathematical preliminaries
- Multiscale methods for EMRIs
- Near small object : $\bar{r} \ll M$
  Puncture [Pound,Miller], multiscale worldline
- Interaction zone: $|r_*| \ll M/\epsilon$
  Multiscale wave equation
- Far zone: $r_* \gg M$
  Geometric optics, with some Post-Minkowski techniques; Extending [Pound 2015]
- **Near-Horizon**: $r_* \ll -M$
  Black hole perturbation theory [Isoyama,Pound,Tanaka,Yamada]
- (future work) Resonances

EMRIs as a weakly nonlinear oscillator

- A two-companion system with $\mu/M \equiv \epsilon \ll 1$
- The general problem:
  - An (instantaneously) nearly geodesic worldline, source scale $\sim \mu$
  - Generates weak metric perturbations $\sim \epsilon$
  - Weak metric perturbations satisfy a weakly nonlinear wave equation, from suppression of quadratic sources
- The motion is non-conservative
  - gravitational radiation carries energy, angular momentum, and Carter to $\mathcal{I}^+, H^+$
- Large separation of timescales $T_{\text{orbit}} \ll T_{\text{RR}}$



Approximating weakly nonlinear differential equations

- The general problem we wish to solve:

$$D[f(t)] + \epsilon Q[f(t)] = 0,$$

with well-understood oscillator differential operator D and nonlinear operator Q.

- A naive approximation takes,

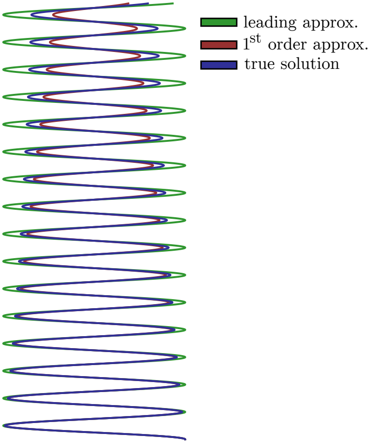
$$f(t) \approx f^{(0)}(t) + \epsilon f^{(1)}(t) + \dots,$$

where

$$D[f^{(0)}(t)] = 0$$

- However, typically D is conservative, while ϵQ introduces dissipation, so at late times,

$$\epsilon f^{(1)} \sim f^{(0)}$$

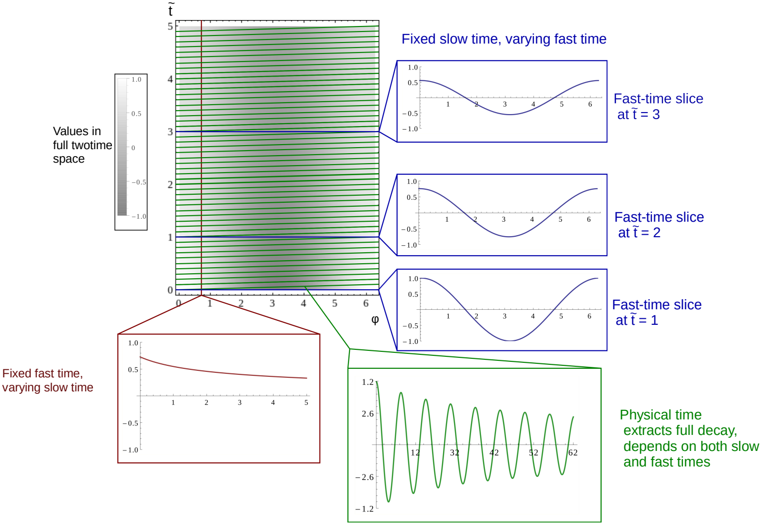


multiscale expanded domain and computation

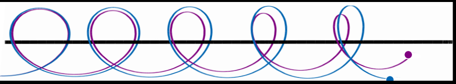
- Dissipation from weak nonlinearity is slow compared to oscillations, $T_{\text{diss}} \sim \epsilon T_{\text{osc}}$
- General procedure:
  - Introduce a pair of time variables $\{\varphi, \tilde{t}\}$, strictly periodic φ, $\tilde{t} = \epsilon t$
  - Promote all physical variables $f(t) \to f'(\varphi, \tilde{t})$
  - Promote all differential operators $D \to D' = D|_{\partial_t \to \Omega(\tilde{t})\partial_\varphi + \epsilon \partial_{\tilde{t}}}$
  - Solve differential equations order-by-order for $f^{(n)}(\varphi, \tilde{t})$ and $\Omega^{(n)}(\tilde{t})$
- Projection to physical solution $f(t)$ via,

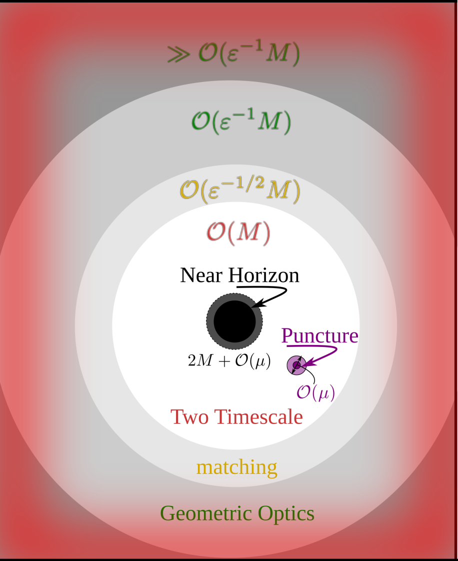
$$\tilde{t} = \epsilon t \qquad \frac{d}{dt}\varphi = \Omega(\tilde{t})$$

multiscale expanded domain and computation



Zones and scales of approximation methods





- Mathematical preliminaries
- Multiscale methods for EMRIs
- Near small object : $\bar{r} \ll M$
  Puncture [Pound,Miller], multiscale worldline
- Interaction zone: $|r_*| \ll M/\epsilon$
  Multiscale wave equation
- Far zone: $r_* \gg M$
  Geometric optics, with some Post-Minkowski techniques; Extending [Pound 2015]
- **Near-Horizon**: $r_* \ll -M$
  Black hole perturbation theory [Isoyama,Pound,Tanaka,Yamada]
- (future work) Resonances

Multiscale approximation construction

Multiscale EMRIs

- Multiscale approximation promotes time dependence to multiple (3 fast, 1 slow) variables $t \to \{\tilde{w}, q^A\}$,

$$w = t + \alpha(r)$$

$$\tilde{w} = \frac{\mu}{M} w \equiv \epsilon w \qquad \frac{d\varphi^A}{dw} = \Omega^A(\tilde{w}, \epsilon)$$

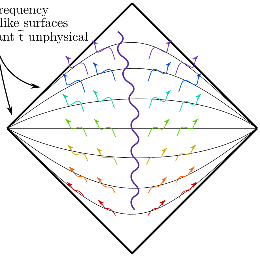
- Action angle variables $\varphi^A = q^A + \mathcal{O}(\epsilon^2)$ used from celestial mechanics solutions
- Metric and worldline ansatz:

$$\begin{aligned} g_{\alpha\beta} =& g^{(0)}_{\alpha\beta}(x^i) + \epsilon h^{(1)}_{\alpha\beta}(\tilde{w}, q^A, x^i) + \epsilon^2 h^{(2)}_{\alpha\beta}(\tilde{w}, q^A, x^i) + \mathcal{O}(\epsilon^3) \\ z^\mu =& z^{(0)}(\tilde{w}, q^A) + \epsilon z^{(1)}(\tilde{w}, q^A) + \mathcal{O}(\epsilon^2) \end{aligned}$$

- Incorporating slow-time evolution $(\tilde{w})$ into the leading solutions preserves quality of approximation for the entire inspiral $\sim M^2/\mu$, and incorporates radiation-reaction worldline into subleading source

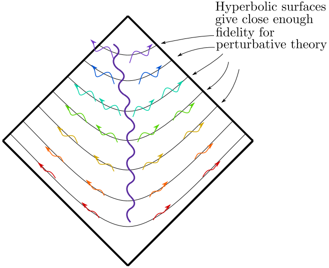
Multiscale EMRIs

- For each slow time, we find the appropriate fast-time solution $h^{(n)}(x^i, \tilde{w}, q^A)$
- True evolution assembled from a path through the 4-dimensional $\{\tilde{w}, q^A\}$
- Quasi-conserved quantities $\{E^{(0)}(\tilde{w}), L_z^{(0)}(\tilde{w}), Q^{(0)}(\tilde{w})\}$ are constant over a surface of constant $\tilde{w}$
  - For spacelike constant $\tilde{w}$ surfaces, transmission of information to distances of $\sim M/\epsilon$ in times of $\sim M$ unphysical
-
  -



Problems at long distances: rapid slow time transmission

- For each slow time, we find the appropriate fast-time solution $h^{(n)}(x^i, \tilde{w}, q^A)$
- True evolution assembled from a path through the 4-dimensional $\{\tilde{w}, q^A\}$
- Quasi-conserved quantities $\{E^{(0)}(\tilde{w}), L_z^{(0)}(\tilde{w}), Q^{(0)}(\tilde{w})\}$ are constant over a surface of constant $\tilde{w}$
  - For spacelike constant $\tilde{w}$ surfaces, transmission of information to distances of $\sim M/\epsilon$ in times of $\sim M$ unphysical
- Solution: enforce surfaces of constant $\tilde{w}$ are asymptotically null
  - Transmission to near-retarded time at $\mathcal{I}^+$ and near-advanced time at H^+ acceptable, approximation convergence restored



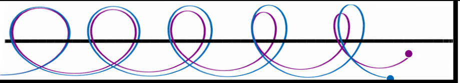
Breakdown at long distances: extended source

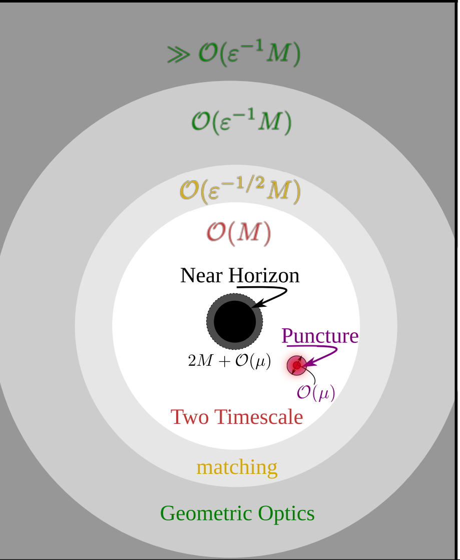
- At large scales of integration domain, another more subtle problem causes a failure of convergence [Pound 2015]
- Multiscale assumes radiation timescale longer than all other time scales
- At each order we solve a wave equation of the form

$$\Box_{q^A} h_{\mu\nu} = S(x^i, q^A, \tilde{w}),$$

  for some source S.
- At long scales, inverting $\Box_{q^A}$ assumes an eternal source (in q^A), so fills space with radiation
- Leading second-order source scales as $\sim \Omega^2/r^2$
  - Leads to a divergent second order solution if taken over full spatial domain
  - Divergence arises even with asymptotically null time variable [Pound 2015]
- A separate approximation is needed for $|r^*| \gg M$

Zones and scales of approximation methods





- Mathematical preliminaries
- Multiscale methods for EMRIs
- Near small object : $\bar{r} \ll M$
  Puncture [Pound,Miller], multiscale worldline
- Interaction zone: $|r_*| \ll M/\epsilon$
  Multiscale wave equation
- Far zone: $r_* \gg M$
  Geometric optics, with some Post-Minkowski techniques; Extending [Pound 2015]
- **Near-Horizon**: $r_* \ll -M$
  Black hole perturbation theory [Isoyama,Pound,Tanaka,Yamada]
- (future work) Resonances

- Perform an action-angle variable decomposition for each fixed $\tilde{w}$, including source terms determined by slow time derivatives
- Forcing terms are determined form self-acceleration as

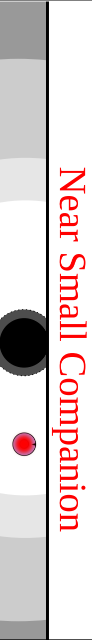
$$g^A = \frac{\partial q^A}{\partial p^\mu} a^\mu$$
$$F^M{}_\mu = \frac{\partial J^M}{\partial P^N} \frac{\partial P^N}{\partial p^\mu} a^\mu,$$

- Finally, action-angle variables obey the multiscale equations:

$$\frac{dq^A}{dw} = \Omega^A = \omega^A[P^{(0)M}(\tilde{w}) + \epsilon P^{(1)M}(\tilde{w}, q^A) + \dots] + \epsilon g^{(1)A}(q^A, P^M) + \epsilon^2 g^{(2)A}(q^A, P^M) + \mathcal{O}(\epsilon^3)$$

$$\frac{dJ^M}{dw} = \epsilon G^{(1)M}(q^A, P^M) + \epsilon^2 G^{(2)M}(q^A, P^M) + \mathcal{O}(\epsilon^3)$$

Near-identity transformations



- The idea: perform a small alteration to the action-angle $\{q^A, J^M\}$ to simplify the equations of motion
- Recently shown to have significant practical importance for rapid computations [Van de Meent, Warburton 2018] **See Niels' talk next**
- Can be used to entirely eliminate [Flanagan, Vines] the angle-variable dependence of self force terms,

$$J'^M = J^M + \epsilon \frac{i\tilde{G}^M_{k^A}}{k^A \Omega_A}$$
$$q'^A = q^A + \epsilon \frac{i}{k^A \Omega_A}\left(\tilde{g}^A_{k^A} - \frac{\partial \omega^A}{\partial J^M} T^M_{k^A}\right)$$

- Resulting equations of motion have only zero-frequency forcing terms,

$$\frac{\partial q'^A}{\partial w} = \omega^A(P^M) + \epsilon g'^{(1)A}(J'^M) + \epsilon^2 g'^{(2)A}(J'^M)$$
$$\frac{\partial J'^M}{\partial w} = \epsilon G'^{(1)}(J'^M) + \epsilon^2 G'^{(2)}(J'^M)$$

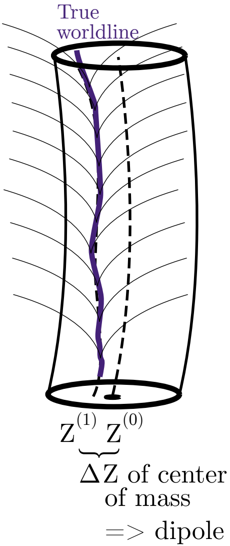
Puncture correction from subleading worldline

Near Small Companion

- Require the puncture metric $h^{\mathcal{P}}_{\alpha\beta}$ through second order
- Formulated in [Pound,Miller] in terms of distance to exact worldline $h^{\mathcal{P}}(z)$
- Instead, for two timescale, worldline is perturbatively expanded

$$z^\mu = z^{(0)\mu}(q^A, \tilde{w}) + \epsilon z^{(1)\mu}(q^A, \tilde{w}) + \mathcal{O}(\epsilon^2)$$

- Gives an $\mathcal{O}(\mu)$ displacement from fiducial worldline $\Rightarrow$ dipole correction
- Expansion of covariant puncture accomplished via techniques presented in [Pound 2015], adjusted to coordinate multiscale time derivatives



Mapping back to worldline expansion

Near Small Companion

- Corrections to puncture require an explicit form of $z^{(1)}(q'^A, J'^M)$ not explicitly given in action-angle equations of motion
- To obtain this inversion, we perturbatively expand

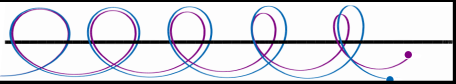
$$\frac{dz^i}{dw} = \frac{p_\beta g^{i\beta}}{p_\beta g^{w\beta}}$$

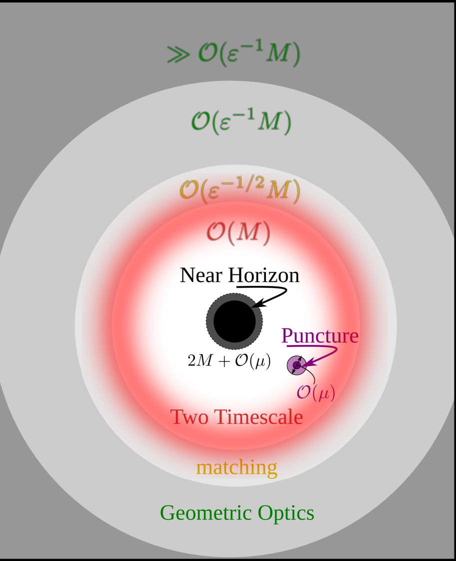
- Requires information from leading and subleading frequencies $\Omega^{(0)}$, $\Omega^{(1)}$
  - Subleading frequencies include self-force contributions $\langle g^A \rangle$
  - Action variable frequencies $\partial H/\partial J^A \equiv \omega^A$ determined by inverting

$$\frac{\partial J^\alpha}{\partial P^\beta}\frac{\partial P^\gamma}{\partial J_\alpha} = \delta^\gamma{}_\beta$$

- Oscillatory dependence of self forces g^A and G^M must be restored in order to obtain full fast-time orbits - all $p^{(1)}$, $\Omega^{(1)}$ depend explicitly on both J'^M, q'^A, g^A, G^M directly

Zones and scales of approximation methods





- Mathematical preliminaries
- Multiscale methods for EMRIs
- Near small object : $\bar{r} \ll M$
  Puncture [Pound,Miller], multiscale worldline
- Interaction zone : $|r_*| \ll M/\epsilon$
  Multiscale wave equation
- Far zone: $r_* \gg M$
  Geometric optics, with some Post-Minkowski techniques; Extending [Pound 2015]
- **Near-Horizon**: $r_* \ll -M$
  Black hole perturbation theory [Isoyama,Pound,Tanaka,Yamada]
- (future work) Resonances

Two timescale wave equations for Lorenz gauge

Interaction Zone

- Practical computations using explicit EFE will likely be performed in Lorenz gauge, promoted to multiscale

$$
\begin{aligned}
\nabla^{(0)}_\mu h^{(1)\mu\nu} &= 0 \\
\nabla^{(1)}_\mu h^{(1)\mu\nu} + \nabla^{(0)}_\mu h^{(2)\mu\nu} &= 0
\end{aligned}
$$

- Imposition of Lorenz gauge gives multiscale relaxed EFE expansion

$$
\begin{aligned}
\delta E^{(0)}_{\mu\nu}[h^{\mathcal{R}(1)}] &= -\,\delta E^{(0)}_{\mu\nu}[h^{\mathcal{P}(1)}] + 8\pi \bar{T}_{\mu\nu} \equiv S^{R\,\mathrm{eff}(1)}_{\mu\nu} \\
\delta E^{(0)}_{\mu\nu}[h^{\mathcal{R}(2)}] &= -\,\delta E^{(0)}_{\mu\nu}[h^{\mathcal{P}(2)}] - \delta^2 E^{(0)}_{\mu\nu}[h^{(1)}, h^{(1)}] - \delta E^{(1)}_{\mu\nu}[h^{(1)}] \equiv S^{R\,\mathrm{eff}(2)}_{\mu\nu}
\end{aligned}
$$

- Effective source formalism - **recall talks by Peter Diener, Seth Hopper**
- Puncture metric determined by z^μ expansion, discussed earlier
- Corrections to geodesic motion incorporated via $E^{(1)}$ terms

- Teukolsky-Lousto-Campanelli formalism offers a way of computing Weyl scalars $\psi^{(1)}_{0/4}$, $\psi^{(2)}_{0/4}$ without first finding the respective metric perturbations.
  - first order:

$$W^{(0)}_{+2}[\psi^{(1)}_0] = 4\pi\Sigma\mathcal{T}_{+2}$$
$$W^{(0)}_{-2}[\rho^{-4}\psi^{(1)}_4] = 4\pi\Sigma\mathcal{T}_{-2}$$

  - second order:

$$W^{(0)}_{+2}[\psi^{(2)}_0] = \mathcal{S}_{+2}[h^{(1)}] - W^{(1)}_{+2}[\psi^{(1)}_0]$$
$$W^{(0)}_{-2}[\rho^{-4}\psi^{(2)}_4] = \mathcal{S}_{-2}[h^{(1)}] - W^{(1)}_{-2}[\rho^{-4}\psi^{(1)}_4]$$

- Second-order source depends on all components of $h^{(1)}$ - must be reconstructed
- TLC equations do not explicitly restrict $\ell < 2$, static completion must be inferred [Merlin et. al.]
  - Slow variation can be computed from reconstructed $h^{(1)}$

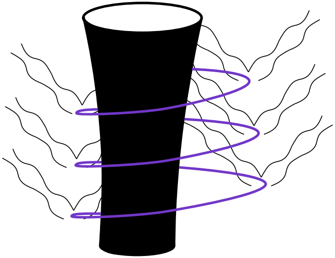
- A sketch of the implementation details, not yet thoroughly developed:
- Need to reconstruct $h^{(1)}$, so prefer leading TLC to be on the physical pointlike source, rather than effective source
  - Sharp feature at all ℓ, at radius of source, require EHS [Barack,Ori,Sago] and transition condition from source **recall from talks by Maarten, Zachary**
  - Static completion part obtained by [Merlin et. al.]
- Expect second-order equations to become ill-defined without regularization, so effective source must be used
- At second order, an extended inhomogeneous source as well as a sharp feature at the radius of the orbit
  - Use extended particular solutions [Hopper, Evans] : separately get regular solution by integrating separation of vars from outside and from inside, transition at sharp feature

Slow variations for spacetime

- General scaling: $\mathcal{O}(\epsilon^2)$ flux, $\mathcal{M}/\epsilon$ time
  - $\mathcal{O}(\epsilon M)$ alteration in spacetime moments over time
- leading order must include δM,δa

$$h^{(1)}_{\alpha\beta} = \delta M(\tilde{w})\frac{\partial g_{\alpha\beta}}{\partial M} + \delta(Ma)(\tilde{w})\frac{\partial g_{\alpha\beta}}{\partial(Ma)} + \mathcal{F}_{\alpha\beta}(P^M, q^A)$$

- What about other secular parts, like spin orientation, overall boost, or more subtle 'charges' from BMS:
  - Each of these introduce a slow-time dependent $\delta h^{(1)}_{\alpha\beta}(\tilde{w})$, but each can (at fixed $\tilde{w}$) be removed with a gauge transformation
  - $\Rightarrow$ up to gauge, all of these effects give rise to a non-removable $\delta h^{(2)}_{\alpha\beta}(\tilde{w})$, but that's post-2-adiabatic.



Determining δM, δa in Lorenz gauge

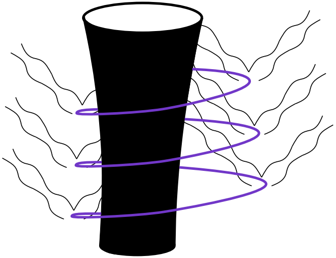
- Consider the additional quasistatic part of the metric perturbation separately, permit a different gauge:

$$
\begin{aligned}
h^{(1)} = &\frac{\partial g^{(0)}}{\partial M}\delta M(\tilde{w}) + \frac{\partial g^{(0)}}{\partial (aM)}\delta(aM)(\tilde{w}) \\
&+ \mathcal{F}^{(1)}(x^i, P^M, q^A)
\end{aligned}
$$

- $E_{\mu\nu}$ annihilates the $\partial g^{(0)}$ parts of variation
- In the multiscale Lorenz gauge, the gauge condition becomes dynamical

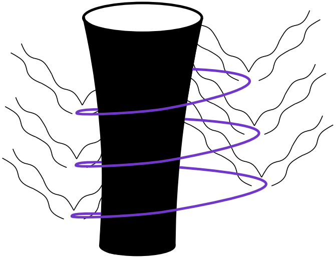
$$
\nabla^{(1)}_{\mu} h^{\mathcal{R}(1)\mu\nu} + \nabla^{(0)}_{\mu} h^{\mathcal{R}(2)\mu\nu} = 0
$$

- In a general sense, this condition is the constraint which preserves stress-energy conservation on the long scale of the inspiral



Determining δM, δa in TLC

- Instead of the Lorenz gauge giving conservation information, more generally we need to consult the EFE itself
- The quasistatic $\ell = 0$ part of the EFE determines the slowly varying parts
$$\int d^3q d^2\Omega R^{(1)}_{tr}[h^{(1)}] = \alpha(r)\partial_{\tilde{u}}\delta M + \beta(r)\partial_{\tilde{u}}\delta a(\tilde{u})$$
$$\int d^3q d^2\Omega R^{(1)}_{\phi r}[h^{(1)}] = \gamma(r)\partial_{\tilde{u}}\delta M + \beta(r)\partial_{\tilde{u}}\delta a(\tilde{u})$$
- These can then be inverted with the EFE to obtain formulas in terms of the second-order Ricci $R[h^{(1)}, h^{(1)}]$
- Note that these derivations intuitively require metric reconstruction to determine quadratic 'fluxes'



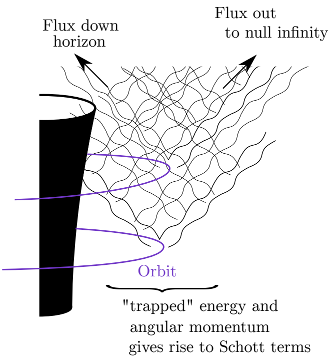
- First order version initially developed by [Gal'tsov; Sago,Fujita; Ganz et. al.]
- At first order, the balance law relations imply give an equality of conserved quantities at the orbit and asymptotic fluxes

$$\left\langle \frac{d\mathcal{E}^{(0)}}{d\tilde{\tau}} \right\rangle = \left\langle u^\alpha u^\beta \mathcal{L}_\xi h^{(1)}_{\alpha\beta} \right\rangle$$

$$\Rightarrow \left\langle \frac{dE}{d\tilde{\tau}} \right\rangle = \sum i\omega \left(\alpha \left(Z^{(1)\text{out}} \right)^2 + \beta \left(Z^{(1)\text{down}} \right)^2 \right)$$

$$\left\langle \frac{dL_z}{d\tilde{\tau}} \right\rangle = \sum im \left(\alpha \left(Z^{(1)\text{out}} \right)^2 + \beta \left(Z^{(1)\text{down}} \right)^2 \right)$$

- A similar identity holds for Carter constant evolution [Mino et. al.]
- At second order, we should anticipate a similar description, but with corrections
  - "Schott" terms from trapped energy in the system



Interaction Zone

- Using the quadratic contribution to second-order self-force [Pound], we derive the second order form of flux balance

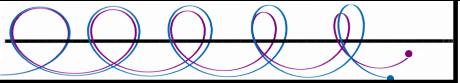
$$\left\langle \frac{d\mathcal{E}^{(1)}}{d\tilde{\tau}} \right\rangle = \frac{1}{2}\left\langle u^\alpha u^\beta \mathcal{L}_\xi h^{(2)}_{\alpha\beta} \right\rangle + \frac{1}{8}\left\langle u^\alpha u^\beta u^\gamma u^\delta \mathcal{L}_\xi \left(h^{(1)\mathcal{R}}_{\alpha\beta} h^{(1)\mathcal{R}}_{\gamma\delta} \right) \right\rangle - \partial_{\tilde{\tau}} \left\langle \xi^\beta u^\gamma h^{(1)\mathcal{R}}_{\beta\gamma} \right\rangle - \frac{1}{2}\partial_{\tilde{\tau}} \left\langle \mathcal{E} u^\alpha u^\beta h^{(1)\mathcal{R}}_{\alpha\beta} \right\rangle$$

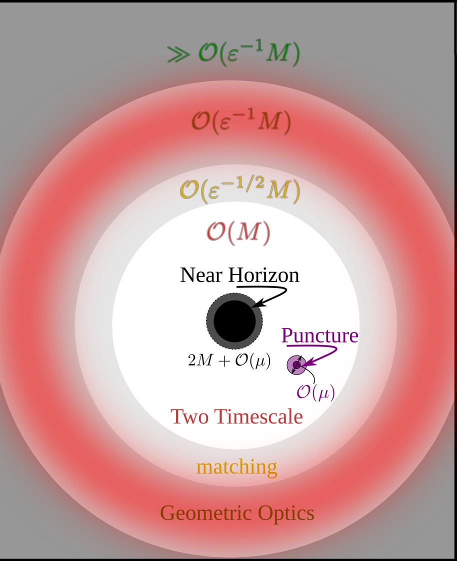
- This is gauge invariant as per full gauge transformation from [Pound '15]
- We are currently developing a version for Carter constant as well
- Additional manipulation expresses this as a sum of contributions:
  - Direct quadratic fluxes from $h^{(2)\mathcal{R}} h^{(1)\mathcal{R}}$ products
  - Corrections associated deviations from homogeneity of $h^{(1)\mathcal{R}}$
  - Integrals over instantaneous in $\tilde{w}$ worldline of
    - Quadratic terms in $h^{(1)}$
    - Terms with $h^{(1)}$ multiplied by gauge vector to Rad. gauge ζ
    - Terms $\sim \partial_{\tilde{w}} h^{(1)}$
    - Terms $\sim \partial_{\tilde{w}} \zeta$

Computational cost?

- A great deal of information can be ‘cached’ by analogy to the osculating geodesics formalism
  - First order solution is a rigorously correct interpolation across instantaneously geodesic solutions
  - The interpolation requires highly accurate frequencies $\Omega(\tilde{w})$, which require second-order solutions
- Second order is a combination of a part sourced also by instantaneously-geodesic contributions and a part which involves explicit time derivatives
- Parameter space : $\{\epsilon, a, \delta M, \delta a, E^{(0)}, E^{(1)}, L_z^{(0)}, L_z^{(1)}, Q^{(0)}, Q^{(1)}\}$
  - Perhaps this seems a bit daunting?
  - Note that the internal spacing in each dimension of leading and subleading parameters does not have to be as small as if we used $E = E^{(0)} + E^{(1)}$, for which we would need spacings $\ll \epsilon\mu$
  - Something I’d be interested in hearing discussion and objections from those that might consider implementations of multiscale

Zones and scales of approximation methods





- Mathematical preliminaries
- Multiscale methods for EMRIs
- Near small object : $\bar{r} \ll M$
  Puncture [Pound,Miller], multiscale worldline
- Interaction zone: $|r_*| \ll M/\epsilon$
  Multiscale wave equation
- Far zone : $r_* \gg M$
  Geometric optics, with some Post-Minkowski techniques; Extending [Pound 2015]
- **Near-Horizon**: $r_* \ll -M$
  Black hole perturbation theory [Isoyama,Pound,Tanaka,Yamada]
- (future work) Resonances

Far Zone

- Spatial scales vary with $\tilde{x}^i \sim \epsilon x^i$, on scale with slow inspiral

- Construct ansatz with single fast variation parameterized by scalar function $\Theta(x^\nu)/\epsilon$

$$g_{\mu\nu}(x^\nu, \varepsilon) = \varepsilon^{-2}\left(\eta_{\mu\nu} + \varepsilon h_{\mu\nu}\left[\tilde{x}^\nu\right] + \varepsilon^2 j_{\mu\nu}\left[\tilde{x}^\nu, \frac{\Theta}{\varepsilon}\right] + \varepsilon^3 k_{\mu\nu}\left[\tilde{x}^\nu, \frac{\Theta}{\varepsilon}\right] + \mathcal{O}(\varepsilon^4)\right)$$

- The rescaling of the coordinates grants an additional order to the weak waves, as they depend on $1/r = \varepsilon/\tilde{r}$

- At leading order, the wave equation for this expansion gives simple $1/\tilde{r}$ radiation dependence

$$\frac{1}{\tilde{r}}\partial_\Theta j_{AB} + \partial_{\tilde{r}}\partial_\Theta j_{AB} = 0$$

- Subleading Lorenz gauge condition constrains additional components of j

- The geometric optics EFE at subleading order fixes the nonvanishing non-TT parts of oscillatory $k_{\mu\nu}$

Far Zone

- Impose Lorenz gauge on the quasistatic part j_0
- Background correction + General wave equation

$$\Box j_{0\mu\nu}[\tilde{x}^\nu] + R_\mu{}^\sigma{}_\nu{}^\rho j_{0\sigma\rho} = -\left\langle G^{(2,2)}_{\mu\nu}[j,j]\right\rangle$$

- Solvable via techniques first introduced by [Blanchet and Damour]
- Particular retarded solution written as integral:

$$j_0 = \mathsf{FP}_{B\to 0}\left[\frac{1}{K(B)}\int_{\tilde{r}}^{\infty} d\tilde{z}\,\frac{S^{(k)}(\tilde{t}-\tilde{z})}{\tilde{r}^k}\,\hat{\tilde{\partial}}_L\left(\frac{(\tilde{z}-\tilde{r})^{B-k+l+2}-(\tilde{z}+\tilde{r})^{B-k+l+2}}{\tilde{r}}\right)\right]$$

- With some manipulation, we can re-write the retarded solution as a further split of homogeneous + particular solution

$$j_{0,\ell} = \tilde{\partial}_L\frac{j^G_\ell(u)}{\tilde{r}} + j^H_\ell(u)$$

- Quasistatic j match inward to the interaction zone to inform quasistatic mode boundary conditions

(Very) Far Zone

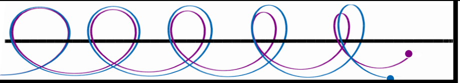
- Evaluate integral assuming large $\tilde{r}$. Geometric optics construction gives $G^{(2,2)} \sim \tilde{r}^{-2}$

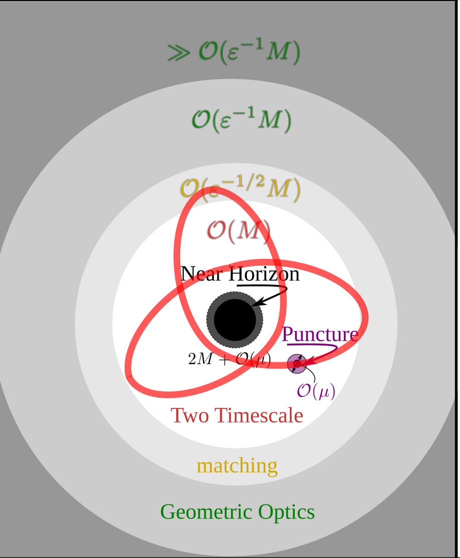
$$j_\ell^H = \frac{\hat{n}_L}{\tilde{r}} \int_0^\infty d\tilde{z} \left(\frac{1}{2} \, \ln \frac{\tilde{z}}{2\tilde{r}} + \sum_{n=1}^{\ell} \frac{1}{n} \right) \left\langle G^{(2,2)}[j,j] \right\rangle + \mathcal{O}(\tilde{r}^{-2} \ln(\tilde{r}))$$

$$\tilde{\partial}_L \frac{j_\ell^G(\tilde{u})}{\tilde{r}} = \tilde{\partial}_L \frac{1}{\tilde{r} K_k} \int_{-\infty}^{\tilde{u}} d\tilde{s} \left\langle G^{(2,2)} \right\rangle (\tilde{s}) (\tilde{u} - \tilde{s})^\ell$$

- Scales similarly with ε to outgoing waves - ‘memory’-like effect
- Scaled coordinates $\tilde{x}$ explicitly incorporate the long scale dependence of the system
  - Region of nonlinear source $r \sim M/\varepsilon \Rightarrow \tilde{r} \sim M$

Zones and scales of approximation methods

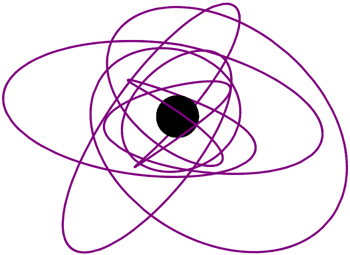




- Mathematical preliminaries
- Multiscale methods for EMRIs
- Near small object : $\bar{r} \ll M$
  Puncture [Pound,Miller], multiscale worldline
- Interaction zone : $|r_*| \ll M/\epsilon$
  Multiscale wave equation
- Far zone: $r_* \gg M$
  Geometric optics, with some Post-Minkowski techniques; Extending [Pound 2015]
- **Near-Horizon**: $r_* \ll -M$
  Black hole perturbation theory [Isoyama,Pound,Tanaka,Yamada]
- (future work) Resonances

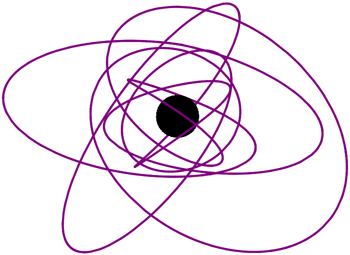
Resonances: They're trouble

- A *resonant* orbit is one in which two characteristic frequencies (e.g. Ω^r and Ω^θ) are related by a rational value
- In the multiscale formalism, resonances cause orbit-averages to develop $\mathcal{O}(1)$ corrections
- Scaling arguments indicate that the duration of the resonant alteration should be $\sim M/\sqrt{\epsilon}$
- Over the course of the resonance, the orbit obtains a phase correction $\sim 1/\sqrt{\epsilon}$ and a geodesic parameter P^M correction of $\sim \sqrt{\epsilon}\mu$ [Flanagan,Hinderer 2010]
    - Significant phase errors will result from ignoring a resonance should it be present



Resonances

- Low-order resonances occur frequently in the geodesic parameter space, particularly dense near the ISCO [Brink, Geyer, Hinderer 2015]
- As shown by a study by [Ruangsri, Hughes 2014]
  - low-order resonances are ubiquitous in parameter space
  - The 3:1 resonance, very close to the ISCO occurred for *all* cases examined
- Order of resonance enters scale of effect as $\Delta\varphi \sim 1/\sqrt{(n+m)\epsilon}$
  - We may need to track resonances to order $|n| + |m| \sim \log(\epsilon)$ [From general scaling arguments from Arnold et. al.]



Resonances

- Multiscale is not invalidated in the case of a resonance, just in need of correction
- Before and after the resonance, the standard non-resonant strategy holds, but needs $\sqrt{\epsilon}$ scale terms

$$P^M = P^{(0)M} + \sqrt{\epsilon}P^{(1/2)M} + \epsilon P^{(1)M} + \mathcal{O}(\epsilon^{3/2})$$
$$q^A = \frac{1}{\epsilon}\left(q^{(0)A} + \sqrt{\epsilon}q^{(1/2)A} + \epsilon q^{(1)A} + \mathcal{O}(\epsilon^{3/2})\right)$$

- We will also need the ‘jumps’ across the resonances in the phase and geodesic parameters [see computation by Van de Meent 2014]
- In general, during the transient resonance, a third time scale emerges $\hat{t} \sim \sqrt{\epsilon}t$, and the dynamics can be computed using a multiscale expansion

Multiscale tapestry approximations

- We now have a nearly complete, comprehensive framework for multiscale approximations
- The description of the interaction zone and far zone are now well-understood
  - We are currently working steadily towards publication of a complete (hopefully implementation-friendly) description
- Several methods work in concert to form a globally valid approximation scheme
- Multiscale approximations are the only current technique which hold the promise to capture *all* post-adiabatic effects consistently
- Future work for multiscale
  - resonances: generally introduce powers $\epsilon^{1/2}$