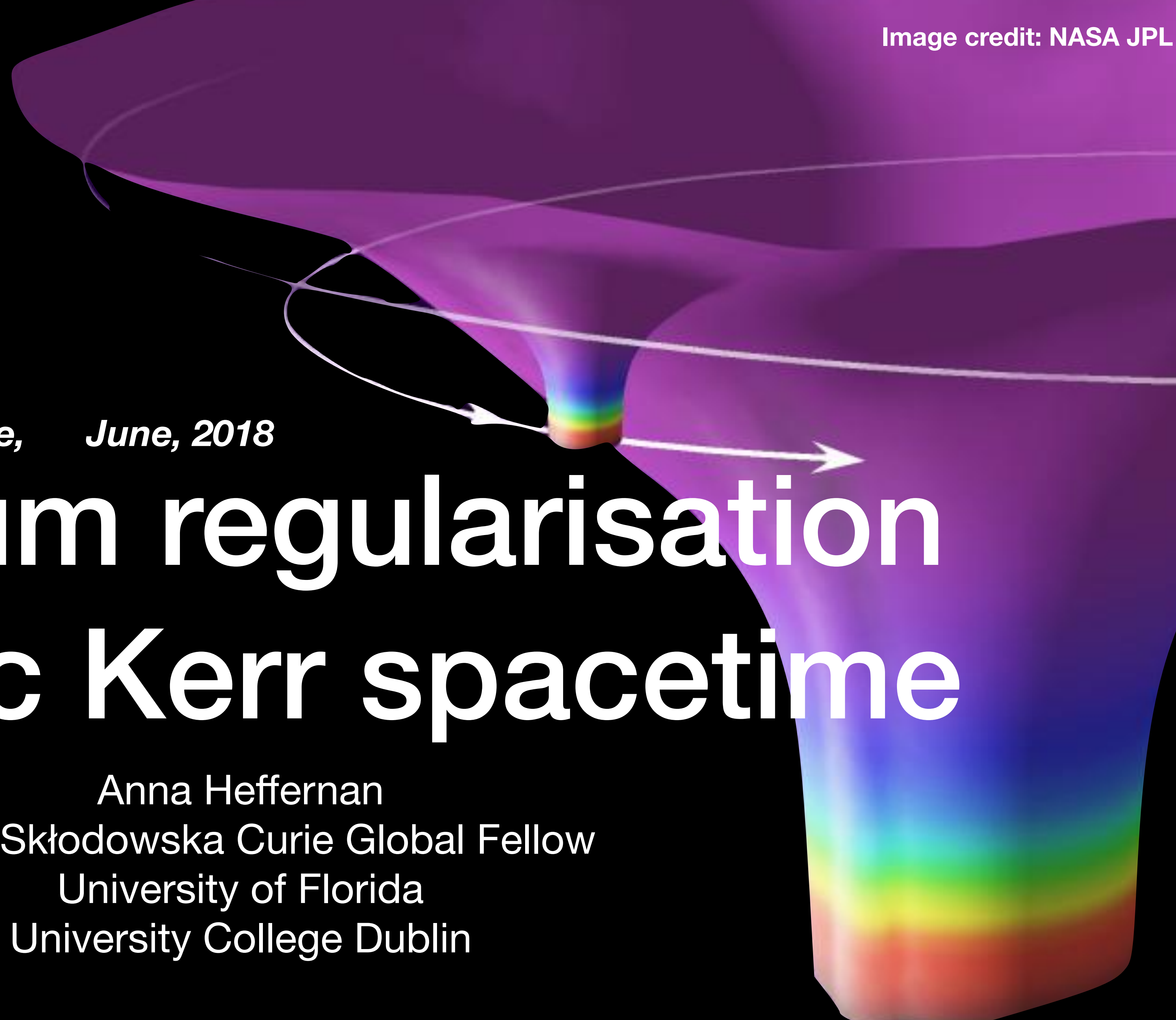


Image credit: NASA JPL

CAPRA21, Albert Einstein Institute, June, 2018

Mode-sum regularisation in generic Kerr spacetime

Anna Heffernan
Marie Skłodowska Curie Global Fellow
University of Florida
University College Dublin

Outline

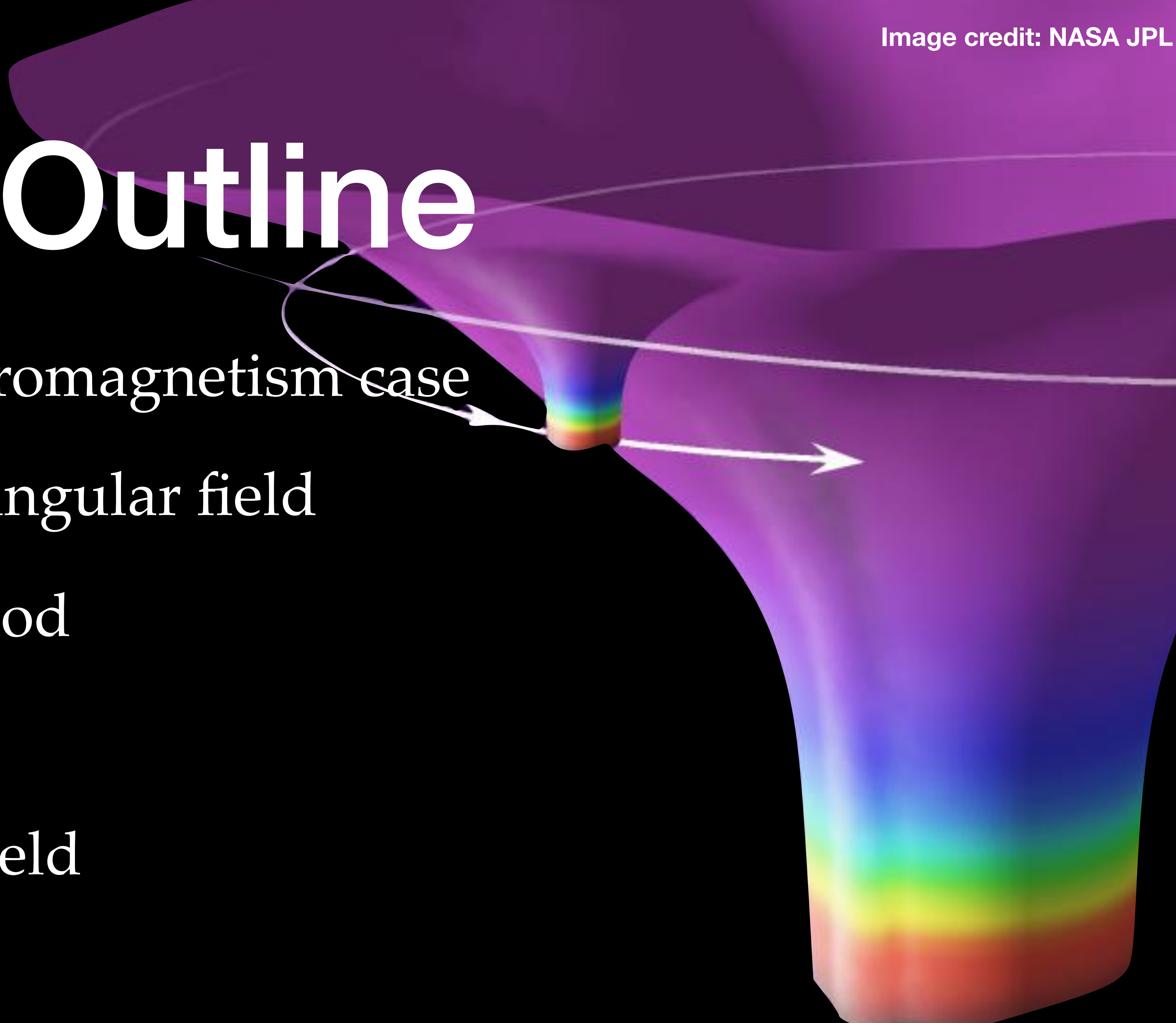


Image credit: NASA JPL

- Flat spacetime: Electromagnetism case
- Detweiler-Whiting singular field
- The mode-sum method
- To date
- Covariant singular field
- New rotation
- Testing
- Road to Gravity

Flat-spacetime

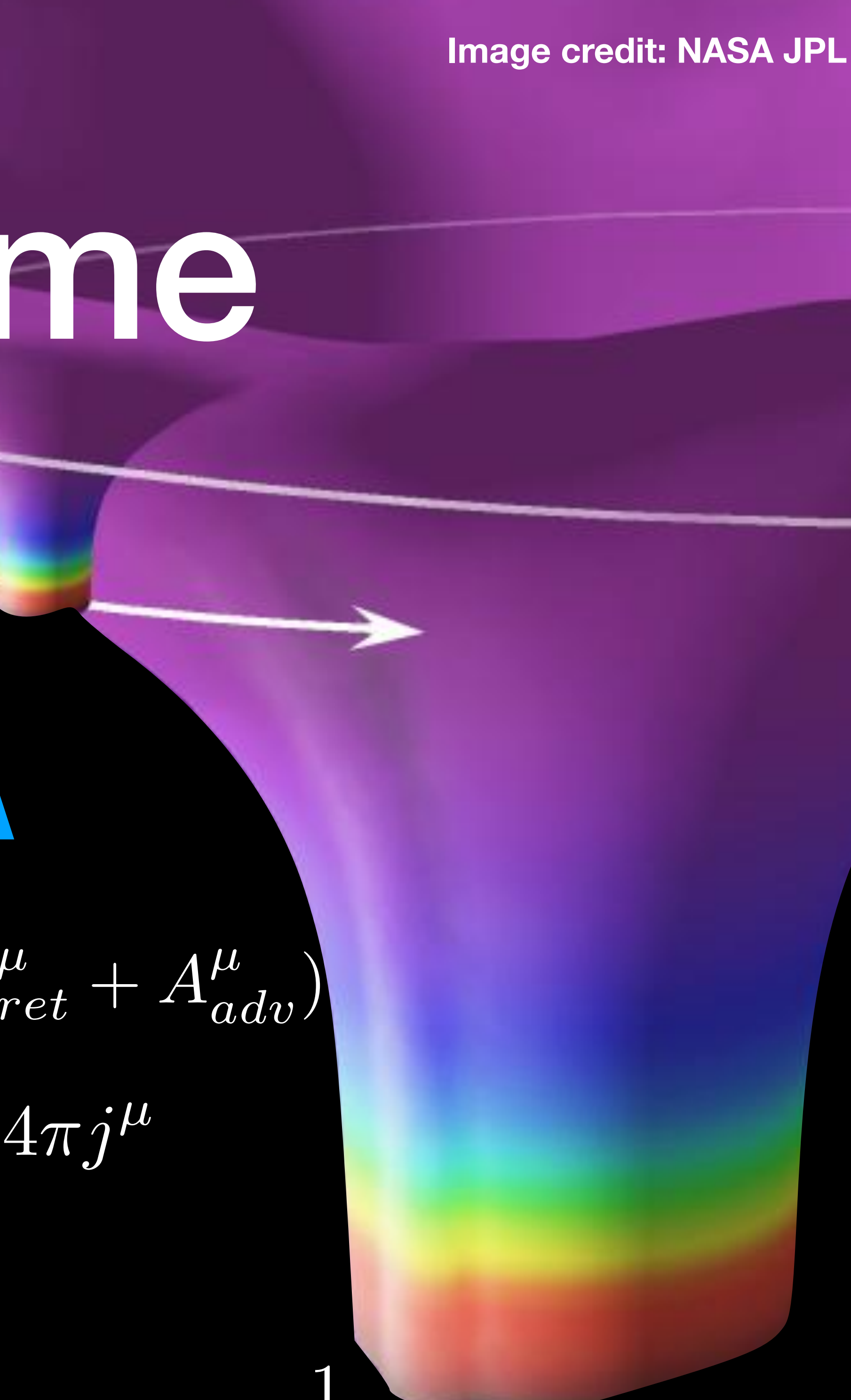
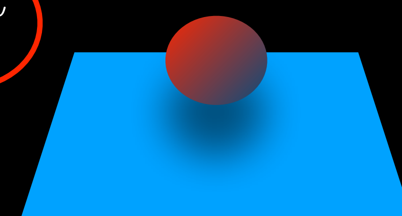
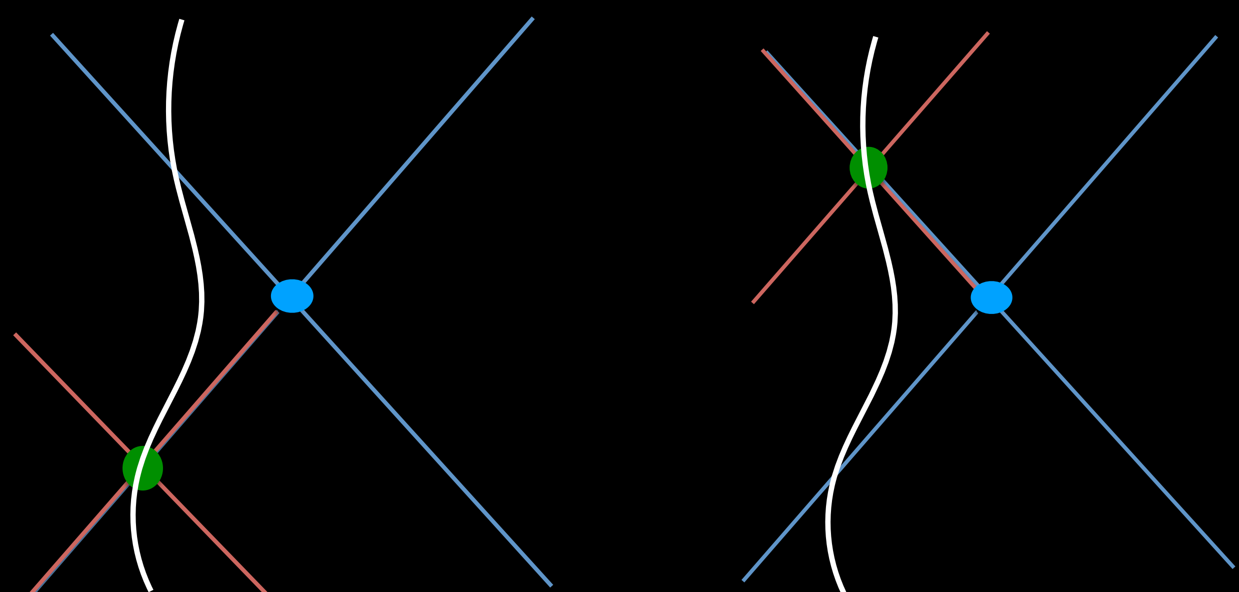


Image credit: NASA JPL

- Flat space
  - Electromagnetism $\Box A^\mu = -4\pi j^\mu$
  - 2 Solutions: A^μ_{ret}, A^μ_{adv} singular!





Retarded solution **Advanced solution**

$$A^\mu_S = \frac{1}{2}\left(A^\mu_{ret} + A^\mu_{adv}\right)$$

$$\Box A^\mu_S = -4\pi j^\mu$$

$$\Box A^\mu_R = 0$$

$$A^\mu_R = A^\mu_{ret} - A^\mu_S = \frac{1}{2}\left(A^\mu_{ret} - A^\mu_{adv}\right)$$

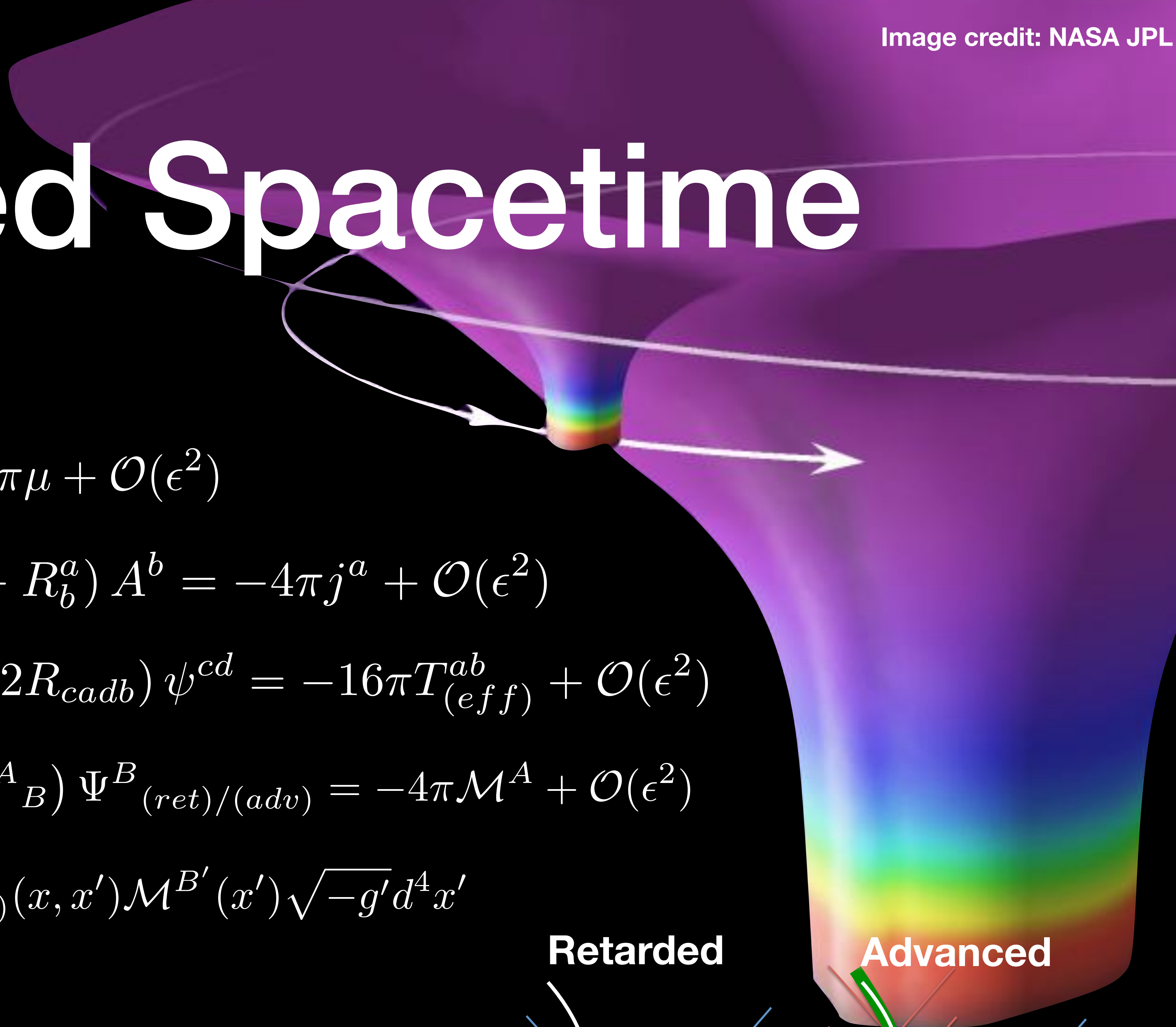
$$F^R_{\mu\nu} = \partial_\mu A^R_\nu - \partial_\nu A^R_\mu$$

$$\rightarrow \quad m a_\mu = f^{ext}_\mu + e F^R_{\mu\nu} u^\nu$$

Dirac, 1938

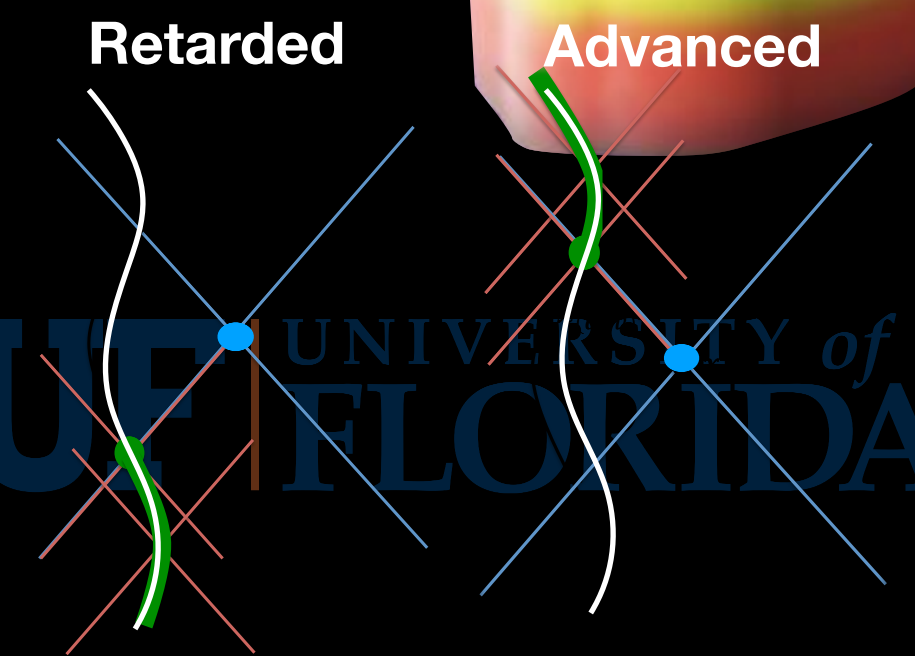
Image credit: NASA JPL

Curved Spacetime

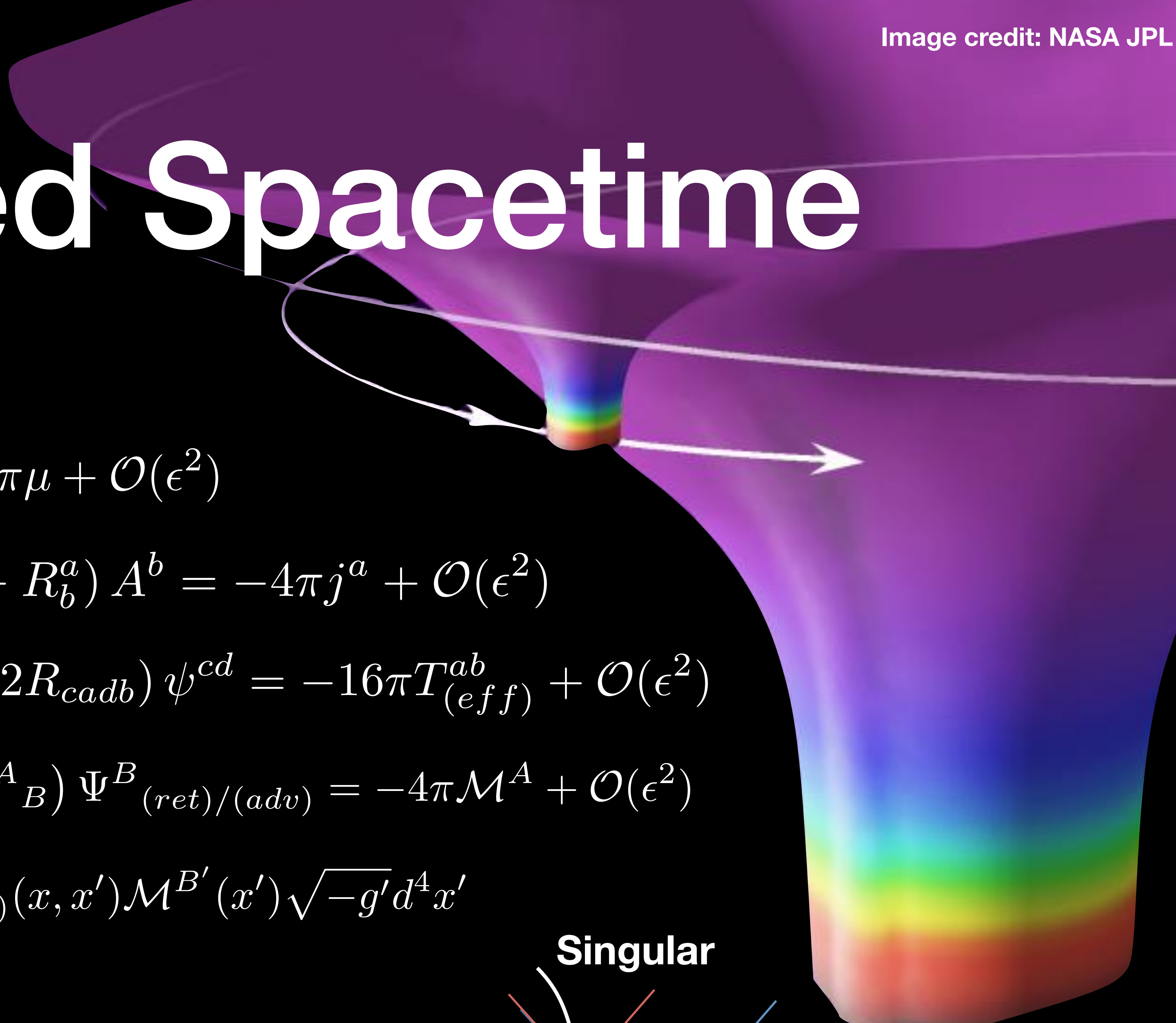


- Curved spacetime
  - Scalar: $(\Box - \zeta R)\,\Phi = -4\pi\mu + \mathcal{O}(\epsilon^2)$
  - Electromagnetic: $(\delta^a{}_b\Box - R^a_b)\,A^b = -4\pi j^a + \mathcal{O}(\epsilon^2)$
  - Gravitational: $(\delta_{ac}\delta_{bd}\Box + 2R_{cadb})\,\psi^{cd} = -16\pi T^{ab}_{(eff)} + \mathcal{O}(\epsilon^2)$
  - General case: $(\delta^A{}_B\Box - P^A{}_B)\,\Psi^B{}_{(ret)/(adv)} = -4\pi\mathcal{M}^A + \mathcal{O}(\epsilon^2)$

$$\Psi^A_{(ret)/(adv)} = \int G^A{}_{B'(ret)/(adv)}(x,x')\mathcal{M}^{B'}(x')\sqrt{-g'}d^4x'$$



Curved Spacetime



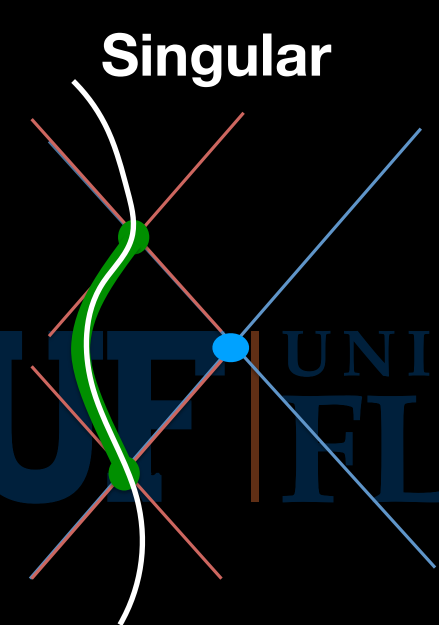
- Curved spacetime
  - Scalar: $(\Box - \zeta R)\,\Phi = -4\pi\mu + \mathcal{O}(\epsilon^2)$
  - Electromagnetic: $(\delta^a{}_b \Box - R^a_b)\,A^b = -4\pi j^a + \mathcal{O}(\epsilon^2)$
  - Gravitational: $(\delta_{ac}\delta_{bd}\Box + 2R_{cadb})\,\psi^{cd} = -16\pi T^{ab}_{(eff)} + \mathcal{O}(\epsilon^2)$
  - General case: $(\delta^A{}_B \Box - P^A{}_B)\,\Psi^B{}_{(ret)/(adv)} = -4\pi\mathcal{M}^A + \mathcal{O}(\epsilon^2)$

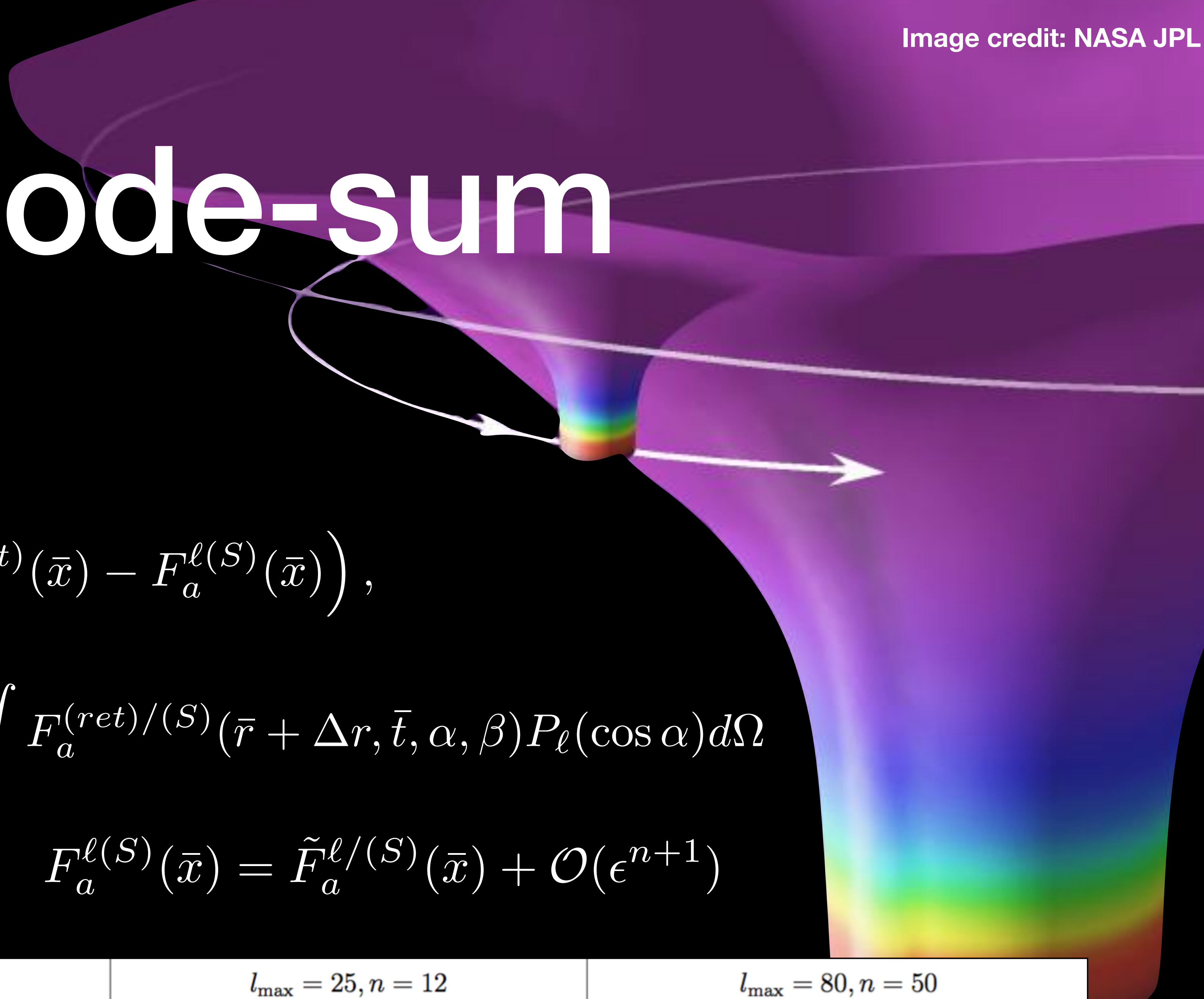
$$\Psi^A_{(ret)/(adv)} = \int G^A{}_{B'(ret)/(adv)}(x,x')\mathcal{M}^{B'}(x')\sqrt{-g'}d^4x'$$

- Detweiler-Whiting singular field

$$G^A{}_{B(S)} = \frac{1}{2}\left[U^A{}_{B'}(x,x')\delta(\sigma) + V^A{}_{B'}(x,x')\Theta(\sigma)\right]$$

$$(\delta^A{}_B \Box - P^A{}_B)\,\Psi^B{}_R = 0, \qquad F^a = p^a{}_A \Psi^A_R$$





Mode-sum

- Mode-sum regularisation
  - Barrack, Ori (2001)

$$F_a(\bar{x}) = \sum_{\ell}^{\infty} \left(F_a^{\ell(ret)}(\bar{x}) - F_a^{\ell(S)}(\bar{x}) \right),$$

$$F_a^{\ell(ret)/(S)}(\bar{x}) = \lim_{\Delta r \to 0} \frac{2\ell+1}{4\pi} \int F_a^{(ret)/(S)}(\bar{r}+\Delta r, \bar{t}, \alpha, \beta) P_\ell(\cos\alpha) d\Omega$$

  - Expansions of the form, $F_a^{\ell(S)}(\bar{x}) = \tilde{F}_a^{\ell/(S)}(\bar{x}) + \mathcal{O}(\epsilon^{n+1})$

→ ℓ^{-n-2} convergence

→ higher order parameters increase efficiency

See: Maarten Van De Meent
Zach Nasipak

	$l_{\max} = 25, n = 12$		$l_{\max} = 80, n = 50$	
RPs used	abs.	rel.	abs.	rel.
AB	$1.3784482573 \times 10^{-5}$	1.2×10^{-10}	$1.37844825756674 \times 10^{-5}$	3.7×10^{-14}
ABD	$1.37844825757 \times 10^{-5}$	5.0×10^{-12}	$1.378448257566791 \times 10^{-5}$	3.3×10^{-15}
$ABDF$	$1.378448257567 \times 10^{-5}$	4.2×10^{-13}	$1.378448257566793 \times 10^{-5}$	1.7×10^{-15}
$ABDFH$	$1.37844825756675 \times 10^{-5}$	3.0×10^{-14}	$1.3784482575667951 \times 10^{-5}$	5.5×10^{-16}
CPU time	155s		4247s	

Data by N.Warburton

Credit: N. Warburton

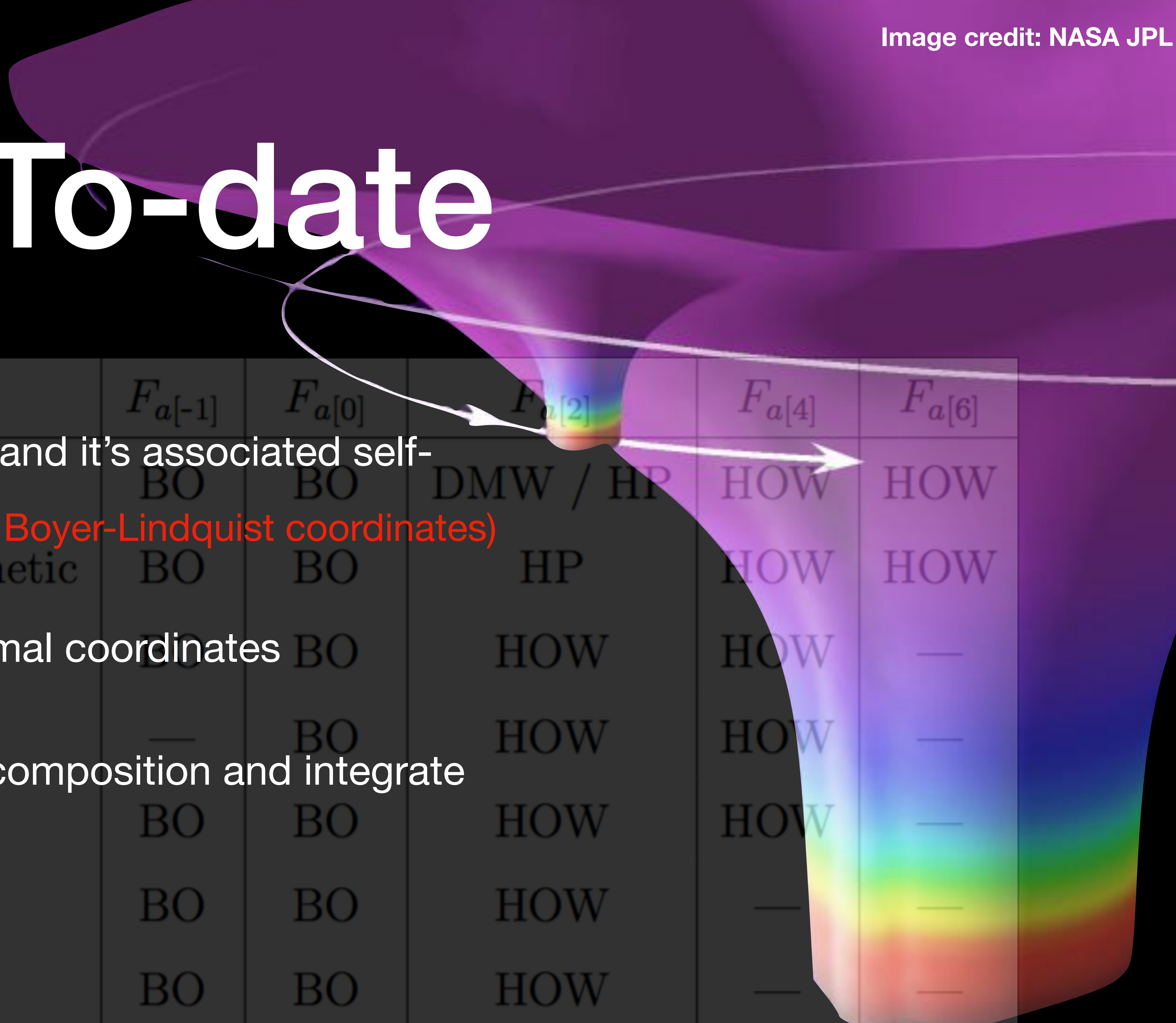
Image credit: NASA JPL

To-date

Cases	$F_{a[-1]}$	$F_{a[0]}$	$F_{a[2]}$	$F_{a[4]}$	$F_{a[6]}$
Schwarzschild scalar	BO	BO	DMW / HP	HOW	HOW
Schwarzschild electromagnetic	BO	BO	HP	HOW	HOW
Schwarzschild gravity	BO	BO	HOW	HOW	—
Schwarzschild *huu*	—	BO	HOW	HOW	—
Kerr scalar	BO	BO	HOW	HOW	—
Kerr electromagnetic	BO	BO	HOW	—	—
Kerr gravity	BO	BO	HOW	—	—
Kerr *huu*	—	HOW	HOW	—	—

BO: Barack, Ori
DMW: Detweiler, Messaritaki, Whiting
HP: Haas, Poisson
HOW: Heffernan, Ottewill, Wardell

To-date



1. Calculate singular field and it's associated self-force (Schwarzschild or Boyer-Lindquist coordinates)
2. Rotate to Riemann normal coordinates
3. Spherical harmonic decomposition and integrate

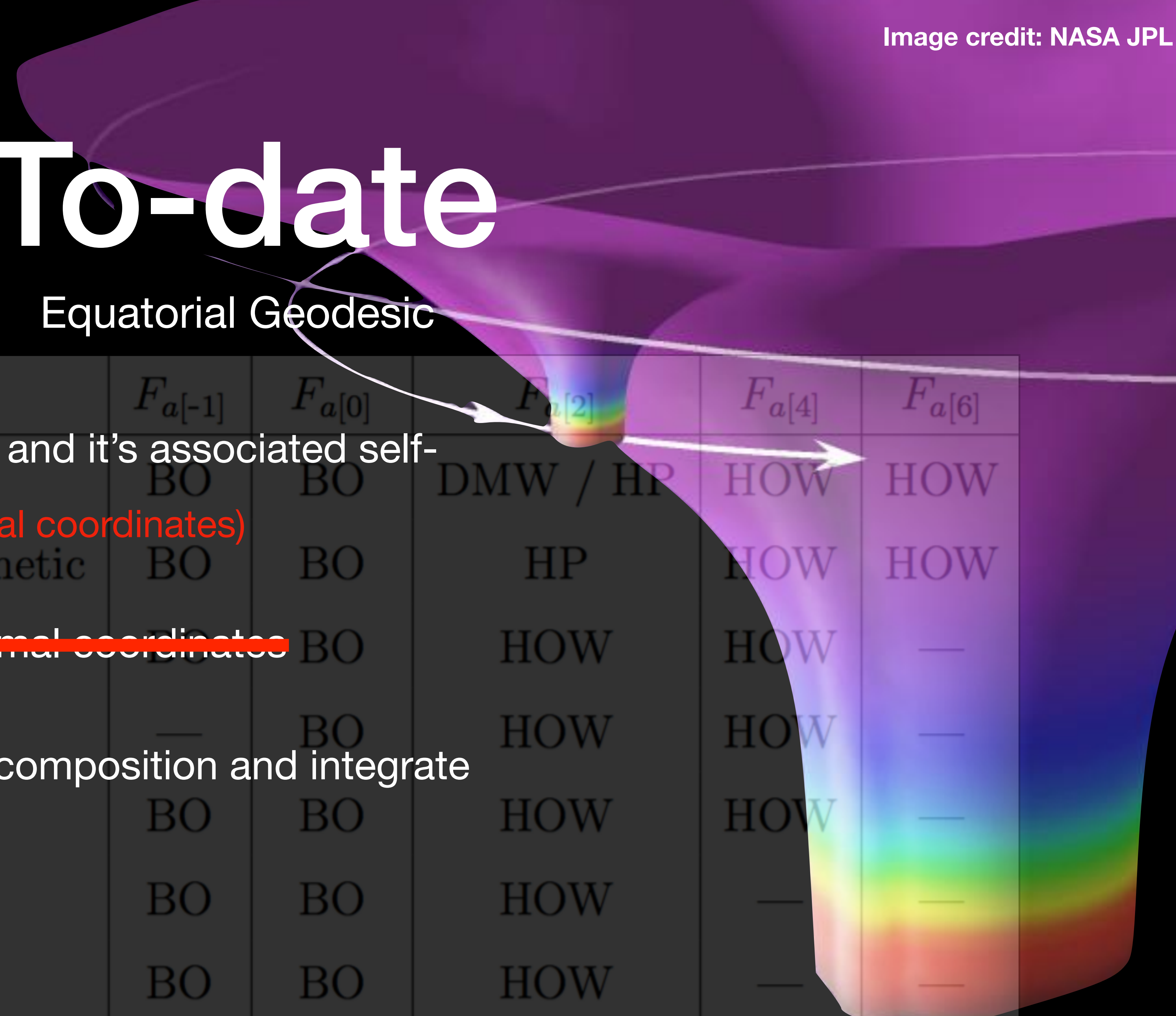
Cases	$F_{a[-1]}$	$F_{a[0]}$	$F_{a[2]}$	$F_{a[4]}$	$F_{a[6]}$
Schwarzschild scalar	BO	BO	DMW / HP	HOW	HOW
Schwarzschild electromagnetic	BO	BO	HP	HOW	HOW
Schwarzschild gravity	BO	BO	HOW	HOW	—
Schwarzschild *huu*	—	BO	HOW	HOW	—
Kerr scalar	BO	BO	HOW	HOW	—
Kerr electromagnetic	BO	BO	HOW	—	—
Kerr gravity	BO	BO	HOW	—	—
Kerr *huu*	—	HOW	HOW	—	—

Riemann Normal Coordinates:

$$w_1 = 2\sin\left(\frac{\alpha}{2}\right)\cos\beta,$$
$$w_2 = 2\sin\left(\frac{\alpha}{2}\right)\sin\beta.$$

$$\sin\theta\cos(\phi - \phi_0') = \cos\alpha,$$
$$\sin\theta\sin(\phi - \phi_0') = \sin\alpha\cos\beta,$$
$$\cos\theta = \sin\alpha\sin\beta.$$

Image credit: NASA JPL



To-date

Equatorial Geodesic

Cases	$F_{a[-1]}$	$F_{a[0]}$	$F_{a[2]}$	$F_{a[4]}$	$F_{a[6]}$
Schwarzschild scalar	BO	BO	DMW / HP	HOW	HOW
Schwarzschild electromagnetic	BO	BO	HP	HOW	HOW
Schwarzschild gravity	BO	BO	HOW	HOW	—
Schwarzschild *huu*	—	BO	HOW	HOW	—
Kerr scalar	BO	BO	HOW	HOW	—
Kerr electromagnetic	BO	BO	HOW	—	—
Kerr gravity	BO	BO	HOW	—	—
Kerr *huu*	—	HOW	HOW	—	—

1. Calculate singular field and it’s associated self-force (In Riemann normal coordinates)

2. ~~Rotate to Riemann normal coordinates~~

3. Spherical harmonic decomposition and integrate

Riemann Normal Coordinates:

$$w_1 = 2\sin\left(\frac{\alpha}{2}\right)\cos\beta,$$

$$w_2 = 2\sin\left(\frac{\alpha}{2}\right)\sin\beta.$$

$$\sin\theta\cos(\phi - \phi_0') = \cos\alpha,$$

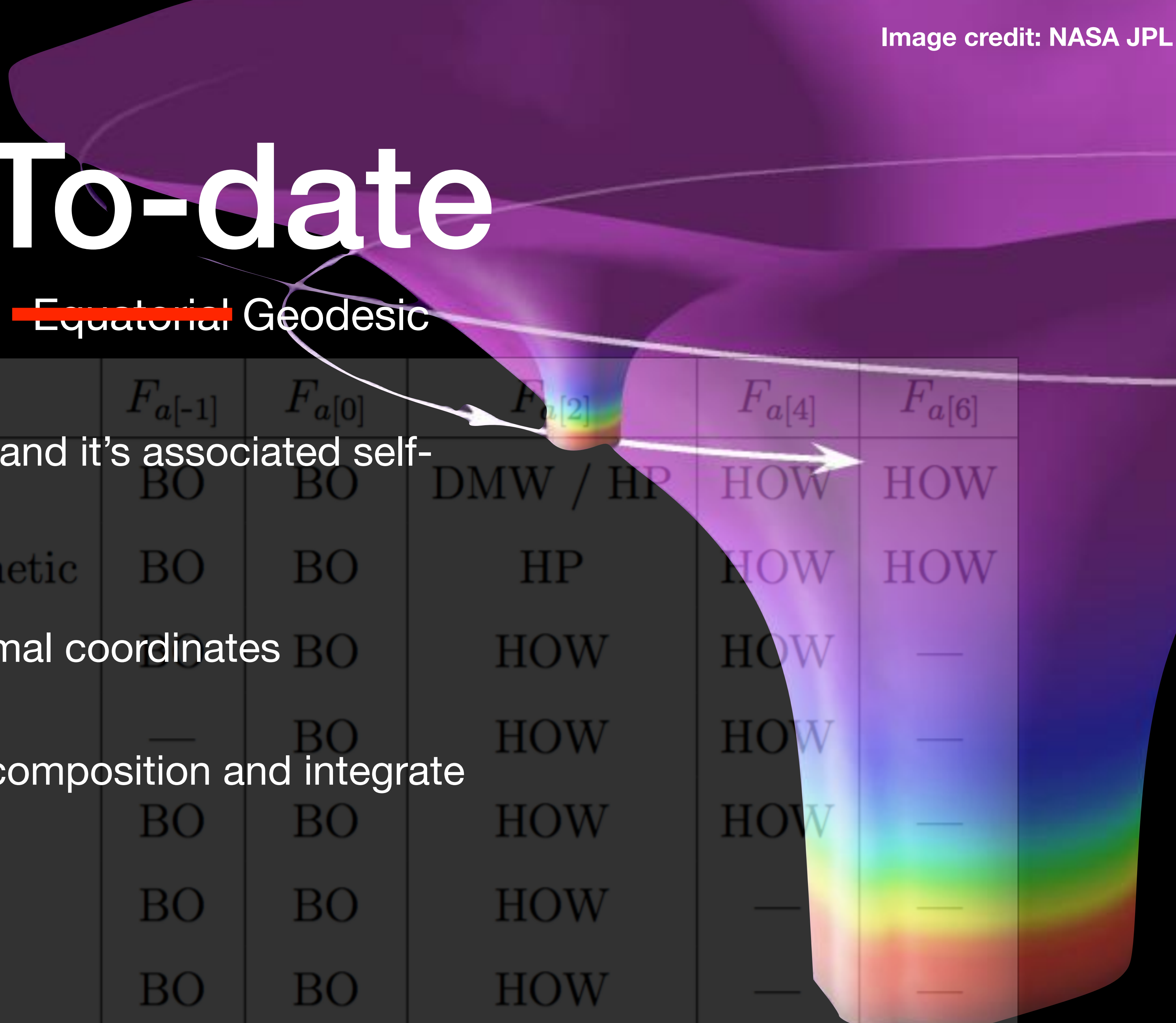
$$\sin\theta\sin(\phi - \phi_0') = \sin\alpha\cos\beta,$$

$$\cos\theta = \sin\alpha\sin\beta.$$

Image credit: NASA JPL

To-date

~~Equatorial~~ Geodesic



1. Calculate singular field and it's associated self-force (covariantly)
2. ~~Rotate to~~ Riemann normal coordinates
3. Spherical harmonic decomposition and integrate

Cases	$F_{a[-1]}$	$F_{a[0]}$	$F_{a[2]}$	$F_{a[4]}$	$F_{a[6]}$
Schwarzschild scalar	BO	BO	DMW / HP	HOW	HOW
Schwarzschild electromagnetic	BO	BO	HP	HOW	HOW
Schwarzschild gravity	BO	BO	HOW	HOW	—
Schwarzschild *huu*	—	BO	HOW	HOW	—
Kerr scalar	BO	BO	HOW	HOW	—
Kerr electromagnetic	BO	BO	HOW	—	—
Kerr gravity	BO	BO	HOW	—	—
Kerr *huu*	—	HOW	HOW	—	—

Riemann Normal Coordinates:

$$w_1 = 2\sin\left(\frac{\alpha}{2}\right)\cos\beta,$$
$$w_2 = 2\sin\left(\frac{\alpha}{2}\right)\sin\beta.$$

$$\sin\theta\cos(\phi-\phi_0') = \cos\alpha,$$
$$\sin\theta\sin(\phi-\phi_0') = \sin\alpha\cos\beta,$$
$$\cos\theta = \sin\alpha\sin\beta.$$

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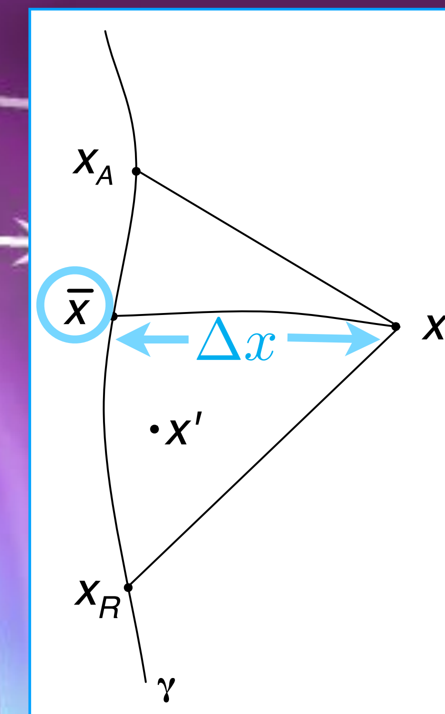
Covariant singular field

- Scalar case

$$\Phi_{(S)}(x) = \frac{q}{2}\left[\frac{U(x,x')}{\sigma_{c'}u^{c'}}\right]^{x'=x_{(adv)}}_{x'=x_{(ret)}} + \cancel{\frac{q}{2}\int_{\tau_{(ret)}}^{\tau_{(adv)}} V(x,z(\tau))d\tau}$$

$$2\sigma = \sigma_a\sigma^a$$



$$\Phi_{[-1]} = \frac{1}{\rho} \qquad \rho^2 = (u_{\bar{a}\bar{b}} + g_{\bar{a}\bar{b}})\Delta x^{ab}$$

$$\Phi_{[0]} = \frac{1}{\rho^3}\left(\frac{1}{2}\bar{A}^b\Delta x_{bcd}\bar{u}^{cd} - \frac{1}{2}\Gamma_{bcd}\Delta x^{bcd} - \frac{1}{2}\Gamma_{cde}\Delta x_b^{de}\bar{u}^{bc}\right) - \frac{1}{\rho}\bar{A}^b\Delta x_b$$

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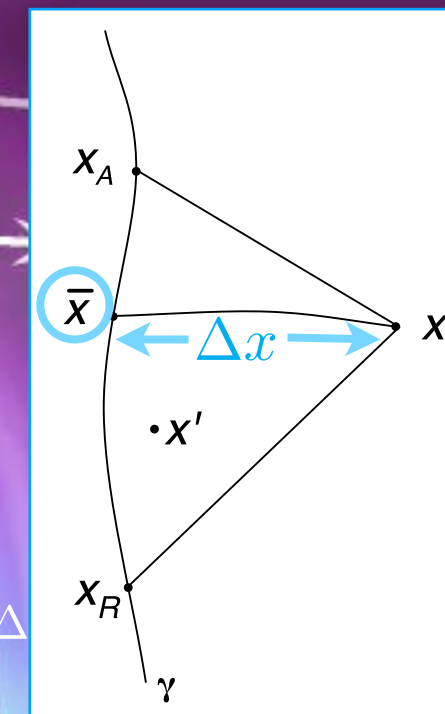
Image credit: NASA JPL

Covariant singular field

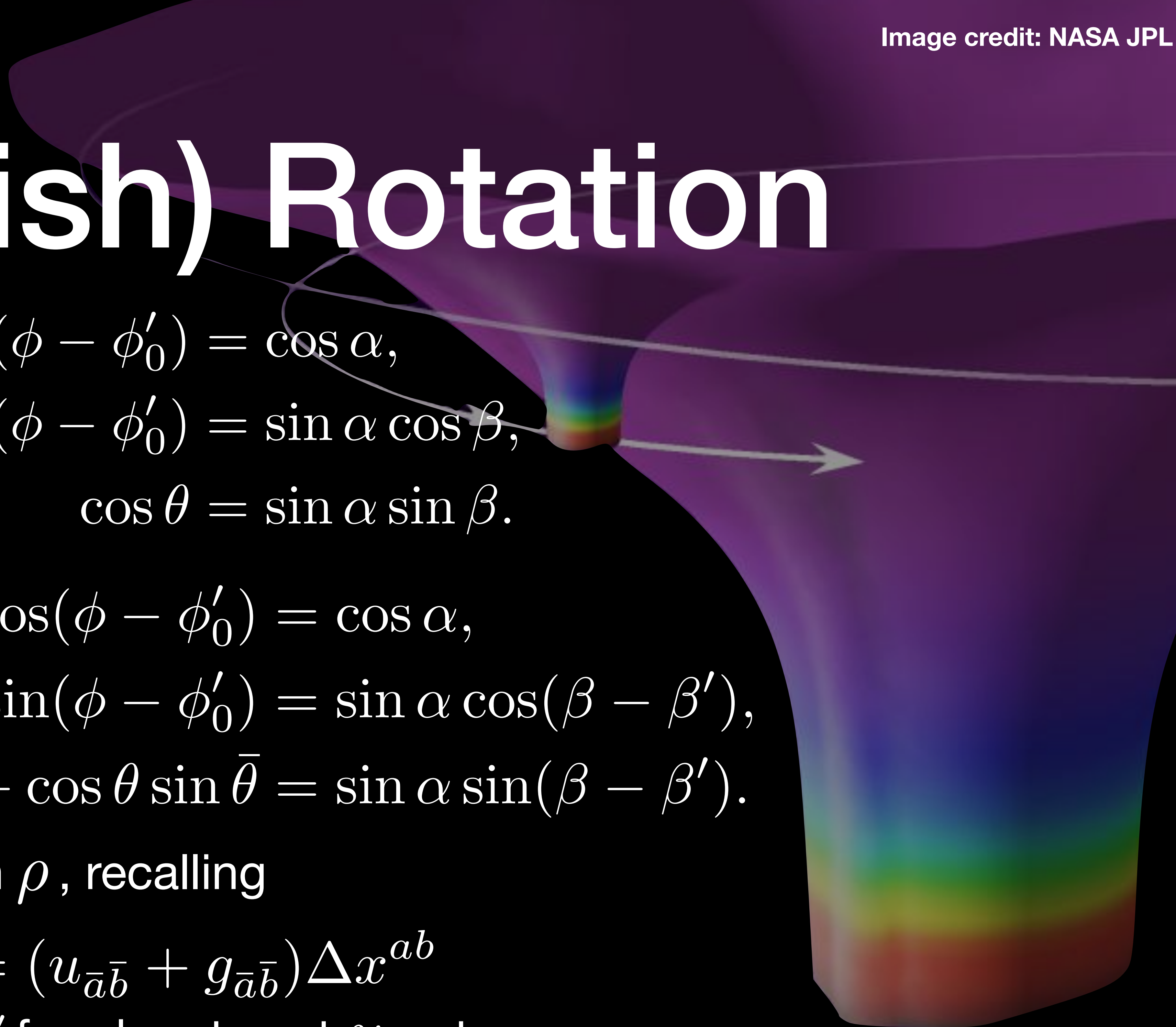
- Scalar case

$$\Phi_{(S)}(x) = \frac{q}{2}\left[\frac{U(x,x')}{\sigma_{c'}u^{c'}}\right]_{x'=x_{(ret)}}^{x'=x_{(adv)}} + \cancel{\frac{q}{2}\int_{\tau_{(ret)}}^{\tau_{(adv)}} V(x,z(\tau))d\tau}$$



$$\begin{aligned}
\Phi_{[1]} = \rho & \left[\frac{1}{2}\dot{\bar{A}}^b\bar{u}_b + \frac{3}{8}\bar{A}_b\bar{A}^b - \frac{1}{12}\Gamma_{fde}\left(\Gamma_{bc}{}^f + \Gamma^f{}_{bc}\right)\bar{u}^{bcde}\right] \\
& + \frac{1}{\rho}\Big[\dot{\bar{A}}^b\left(\bar{u}_b{}^{cd}\Delta x_{cd} + \frac{1}{2}\Delta x_{bc}\bar{u}^c\right) + \frac{1}{6}\left(\Gamma_{bde,c} + 2\Gamma_{dbe,c} - \Gamma_{bcd,e} - 2\Gamma_{dbc,e} + \Gamma^f{}_{bc}\Gamma_{fde} - \Gamma^f{}_{bd}\Gamma_{fce}\right)\bar{u}^{bc}\Delta \\
& + \bar{A}^b\left(\frac{3}{8}\bar{A}^c\Delta x_{bc} + \frac{3}{4}\bar{A}_b\bar{u}^{cd}\Delta x_{cd} - \frac{1}{4}\Delta x^{cd}\Gamma_{bcd}\right) + \frac{1}{4}\left(4\Gamma_{cdf,e} - 4\Gamma_{cde,f} + \Gamma_{fc}{}^i\Gamma_{ide} + \Gamma^i{}_{cd}\Gamma_{ief}\right)\bar{u}^{bcde}\Delta x_b{}^f\Big] \\
& + \frac{1}{\rho^3}\Big\{ -\frac{1}{6}\dot{\bar{A}}^b\bar{u}^{cde}\Delta x_{cde}\left(\Delta x_b + \bar{u}_b{}^f\Delta x_f\right) \\
& - \frac{1}{8}\bar{A}^b\Delta x_{cde}\Big[\bar{u}^{de}\left(6\bar{A}^c\Delta x_b + \bar{A}_b\bar{u}^c\right) + 2\bar{u}^{cd}\Delta x_c\left(3\Gamma_d{}^{ef}\Delta x_b + \Gamma_b{}^{ef}\right) + 2\Gamma_c{}^{de}\Delta x_b\Big] \\
& + \frac{1}{12}\bar{u}^{bcde}\Delta x_{bc}\Big[2\Delta x^{fi}\Big(\Gamma_{jei}\Gamma^j_{df} - \Gamma_{jfi}\Gamma^j_{de} - \Gamma_{dpi,e} - 2\Gamma_{fid,e} + \Gamma_{def,i} + 2\Gamma_{fde,i}\Big) \\
& + \bar{u}^{fi}\Delta x_d\Big(\Gamma_{njk}\Big(\Gamma_{fi}{}^n + \Gamma^n{}_{fi}\Big)\bar{u}^{jk}\Delta x_e + \Delta x^j\Big(\Gamma_{je}{}^k\Gamma_{kfi} + \Gamma_{kij}\Gamma^k{}_{ef} - 4\Gamma_{efj,i} + 4\Gamma_{efi,j}\Big)\Big)\Big] \\
& + \frac{1}{2}\bar{u}^{bc}\Delta x^{def}\left[\frac{1}{3}\Delta x_b\Big(\Gamma_{ief}\Gamma^i{}_{cd} - \Gamma_{def,c} - \Gamma_{cde,f} + \Gamma_{dce,f}\Big) - \frac{1}{4}\Gamma_{bde}\Gamma_{cfi}\Delta x^i\right]\Big\} \\
& + \frac{1}{8\rho^5}\Big\{3\bar{A}^b\bar{u}^d\Delta x_{bcd}\Big[\bar{A}^c\bar{u}^{refi}\Delta x_f - 2\bar{u}^c\Big(\Gamma^{refi}\Delta x_{cf} + \Gamma^f{}_{ij}\bar{u}_{ef}\Delta x^j\Big)\Big] \\
& + 3\Delta x^{bfi}\left(\Gamma_{bcd}\Gamma_{efi}\Delta x^{cde} + 2\Gamma^c{}_{de}\Gamma_{fij}\bar{u}_{bc}\Delta x^{dej}\Gamma^d{}_{fi}\Gamma^e{}_{lk}\bar{u}_{bcde}\Delta x^{cfk} + \right)\Big\}
\end{aligned}$$

Image credit: NASA JPL



New(ish) Rotation

Previous Rotation:

$$\sin\theta\cos(\phi-\phi_0') = \cos\alpha,$$
$$\sin\theta\sin(\phi-\phi_0') = \sin\alpha\cos\beta,$$
$$\cos\theta = \sin\alpha\sin\beta.$$

New Rotation:

$$\cos\theta\cos\bar\theta + \sin\theta\sin\bar\theta\cos(\phi-\phi_0') = \cos\alpha,$$
$$\sin\theta\sin(\phi-\phi_0') = \sin\alpha\cos(\beta-\beta'),$$
$$-\sin\theta\cos\bar\theta\cos(\phi-\phi_0') + \cos\theta\sin\bar\theta = \sin\alpha\sin(\beta-\beta').$$

Target: No Δx cross terms in ρ, recalling

$$\rho^2 = (u_{\bar a\bar b} + g_{\bar a\bar b})\Delta x^{ab}$$

Leor (15+ years ago): Use β' freedom to set $u_{\bar w_2}$ to zero

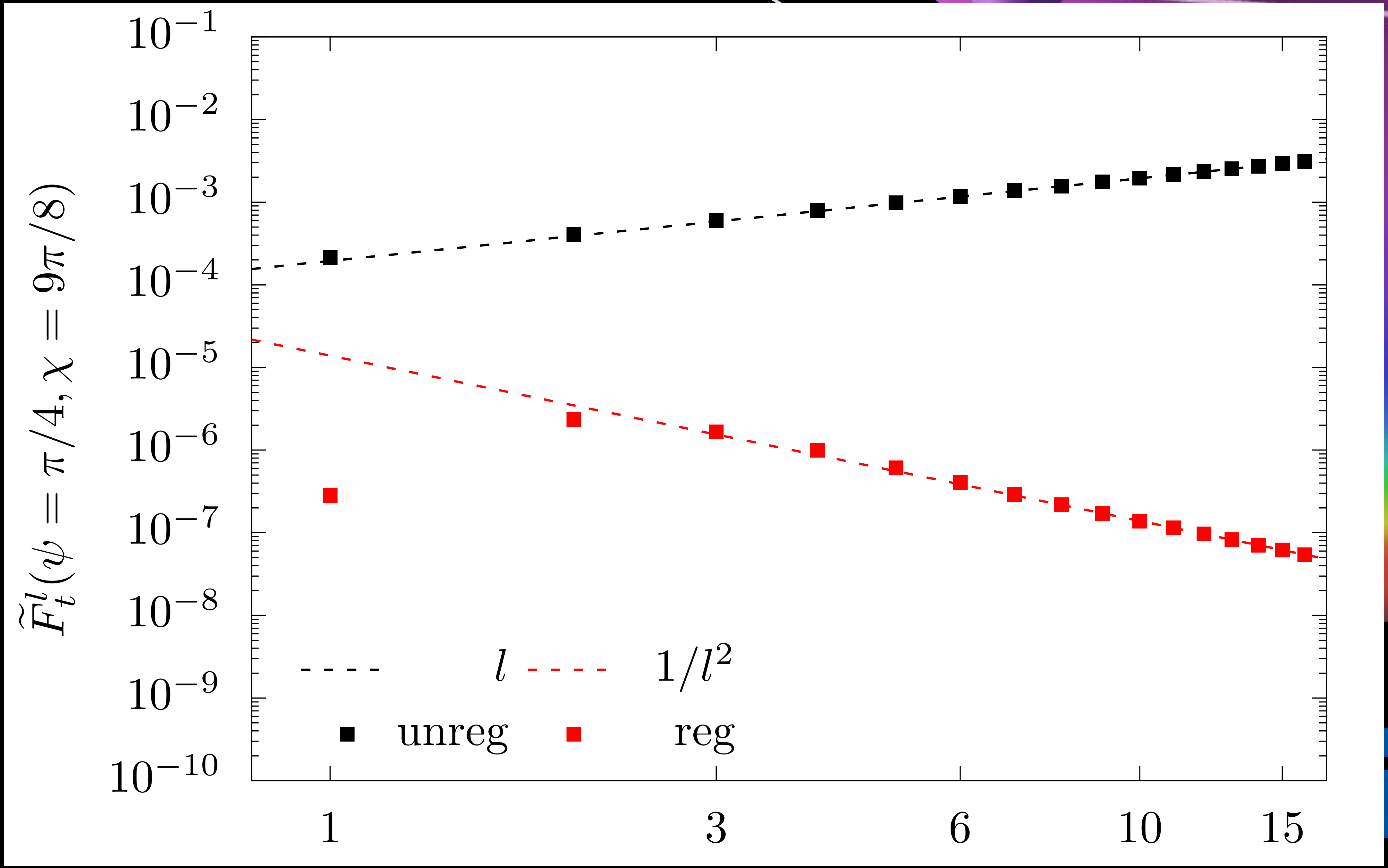
(explicit calculation avoided in Boyer Lindquist)

Explicitly in Riemann normal coordinates: Not viable

Solution: Don't reinvent the wheel! Forget about A parameter!

Image credit: NASA JPL

Testing: Scalar



Data by Zach Nasipak

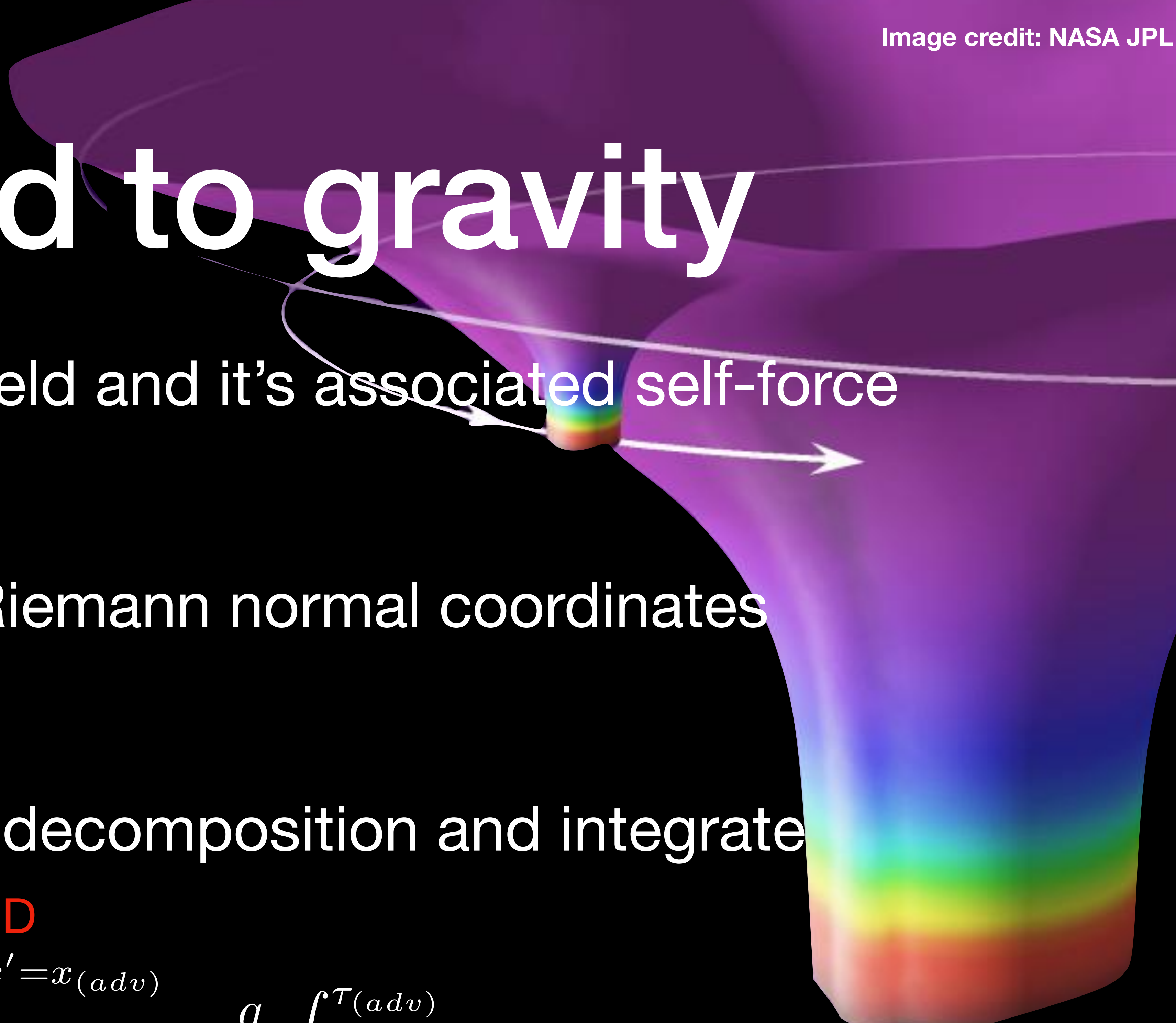


Image credit: NASA JPL

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Road to gravity

1. Calculate singular field and it’s associated self-force
   - Covariant Scalar: D
   - Non-geodesic: D
2. Break covariance: Riemann normal coordinates
   - Schwarzschild: D
   - Kerr: B
3. Spherical harmonic decomposition and integrate
   - Coordinate independent: D

$$\Phi_{(S)}(x) = \frac{q}{2}\left[\frac{U(x,x')}{\sigma_{c'}u^{c'}}\right]_{x'=x_{(ret)}}^{x'=x_{(adv)}} + \frac{q}{2}\int_{\tau_{(ret)}}^{\tau_{(adv)}} V(x,z(\tau))d\tau$$

$$\bar{h}_{ab}^{(S)} = 2\mu\left[\frac{u^{a'}u^{b'}U_{aba'b'}(x,x')}{\sigma_{c'}u^{c'}}\right]_{x'=x_{(ret)}}^{x'=x_{(adv)}} + 2\mu\int_{\tau_{(ret)}}^{\tau_{(adv)}} V_{aba'b'}(x,z(\tau))u^{a'}u^{b'}d\tau.$$

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Image credit: NASA JPL

Thank you!

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