Signatures of Extra Dimensions in Gravitational Waves from Black Hole Quasi-Normal Modes

Sumanta Chakraborty
IACS, Kolkata, India

21st CAPRA Meeting
AEI, Potsdam
26th Jun, 2018
1. Introduction to Background Spacetime.
3. Possible signatures of extra dimensions in the Quasi-Normal Modes.

References

Why Extra Dimensions?

- The basic motivation for existence of extra dimensions is the renormalization of Higgs mass.
- The counter-term needed for mass renormalization corresponds to,

\[ \delta m^2_{H} = \frac{\Lambda^2}{8\pi^2} \left( \lambda_H - \lambda_F^2 \right) + \log \text{ div.} + \text{finite terms} \]

- Since the cutoff scale \( \Lambda \) is in the Planck regime, we must have a fine tuning of the couplings to get renormalized Higgs mass at the Electro-weak scale.
- Extra dimension is one particular method, which was invoked to solve the above issue.
The five dimensional gravitational field equations read

\[ G_{AB} = 8\pi G_{(5)} T_{AB} \]

When the bulk energy momentum tensor is originating from a negative cosmological constant \( \Lambda \), one arrives at the following static and spherically symmetric solution on the brane,

\[ ds^2_{\text{unperturbed}} = e^{-2ky} \left( -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \right) + dy^2 \]
tension = $+48\pi / G \ell^2$

$tension = -48\pi / G \ell^2$

$\mathbb{Z}_2$ symmetry

warped horizon

$r = 0$

black string bulk

$y = 0$

$y = d$

$2G/M$
The normal $n_A = \nabla_A y$, yields the induced metric on the brane hypersurface to be $h_{AB} = g_{AB} - n_A n_B$, such that $n_A h^A_B = 0$.

Effective Field Equations

$^{(4)} G_{\mu\nu} + E_{\mu\nu} = 0$

Here $E_{\mu\nu}$ stands for a particular projection of the bulk Weyl tensor $C_{ABCD}$ on the brane hypersurface.

Weyl Stress

$$E_{\mu\nu} = C_{ABCD} e^A_\mu n^B e^C_\nu n^D$$
Perturbation to first order

- Perturbation of the effective field equations around the bulk metric \( g_{AB} \), such that \( g_{AB} \to g_{AB} + h_{AB} \).
- There are redundant gauge degrees of freedom. The following gauge conditions (known as the Randall-Sundrum gauge)

\[
\nabla_A h^A_B = 0; \quad h^A_A = 0; \quad h_{AB} = h_{\alpha\beta} e^\alpha_A e^\beta_B
\]

- The perturbed bulk metric takes the following form,

\[
ds^2_{\text{perturbed}} = \left[ q_{\alpha\beta}(y, x^\mu) + h_{\alpha\beta}(y, x^\mu) \right] dx^\alpha dx^\beta + dy^2
\]
The imprints of the presence of extra dimensions are through two quantities — (a) Size of the extra dimension $d$ and (b) the bulk curvature scale $\ell = 1/k$.

The dimensionless ratio $d/\ell$ is an important one and if one wishes to solve the hierarchy problem we must have $d/\ell \geq 12$.

The above model can also be written as a Brans-Dicke theory, with the Brans-Dicke parameter $\omega_{bd}(d/\ell)$.

Thus to be consistent with local physics we must have $d/\ell \geq 5$.

Finally, the black hole mass and the bulk curvature scale has to satisfy some constraint to avoid the Gregory-Laflamme instability.
Assuming a separable perturbation

\[ h_{\alpha\beta}(y, x^\mu) = h_{\alpha\beta}(x^\mu) \chi(y) \]

the perturbed effective equations can be decomposed into two parts:

**Separability**

\[
e^{-2ky} \{ -k^2 \chi + 3k \partial_y \chi + \partial^2_y \chi \} = -\mathcal{M}^2 \chi(y)
\]

\[
(4) \Box h_{\mu\nu} + 2h_{\alpha\beta} (4) R_{\alpha\beta}^{\mu\nu} - \mathcal{M}^2 h_{\mu\nu} = 0
\]

With \( \mathcal{M} = 0 \), one immediately recovers the dynamical equation governing gravitational perturbation in general relativity.
The equation for $\chi(y)$ is essentially Bessel’s differential equation and hence it’s two independent solutions are

$$
\chi(y) = e^{-\frac{3}{2}ky} \left[ C_1 J_\nu \left( \frac{me^{ky}}{k} \right) + C_2 Y_\nu \left( \frac{me^{ky}}{k} \right) \right]
$$

The boundary conditions imposed are derivatives of $\chi = 0$ at $y = 0$ and also on $y = d$. This leads to the following algebraic equation

$$
Y_{\nu-1}(m_n/k)J_{\nu-1}(z_n) - J_{\nu-1}(m_n/k)Y_{\nu-1}(z_n) = 0
$$

Here $m_n = \{z_n k\} e^{-kd}$ are Kaluza-Klein mode masses.
In this case there are two master variables, \( u_{n,l} \) and \( v_{n,l} \) respectively and their evolution equations read

\[
\mathcal{D} u_{n,l} + f(r) \left\{ m_n^2 + \frac{l(l+1)}{r^2} - \frac{6}{r^3} \right\} u_{n,l} + f(r) \frac{m_n^2}{r^3} v_{n,l} = 0
\]

\[
\mathcal{D} v_{n,l} + f(r) \left\{ m_n^2 + \frac{l(l+1)}{r^2} \right\} v_{n,l} + 4f(r) u_{n,l} = 0
\]

Here, \( \mathcal{D} \) is the differential operator \( \partial_t^2 - \partial_{r_*}^2 \), where \( r_* \) is the tortoise coordinate defined using \( f(r) \) as \( dr_* = dr/f(r) \).
Table: Imaginary parts of the quasi-normal mode frequencies have been presented for $d/\ell = 20; 1/\ell = 6 \times 10^7$.

<table>
<thead>
<tr>
<th></th>
<th>$m = 0.44, l = 2$</th>
<th>$m = 0.83, l = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Imaginary</td>
<td>-0.051</td>
<td>-0.038</td>
</tr>
<tr>
<td></td>
<td>-0.071</td>
<td>-0.104</td>
</tr>
<tr>
<td></td>
<td>-0.197</td>
<td>-0.168</td>
</tr>
<tr>
<td></td>
<td>-0.239</td>
<td>-0.369</td>
</tr>
</tbody>
</table>
Figure: Time evolution of the master mode function \( u_{n,l}(t) \) associated with axial gravitational perturbation for two different values of angular momentum \( l \) in the context of general relativity have been depicted.
Figure: Time evolution of the master mode function $u_{n,l}(t)$ for general relativity ($n = 0$) as well as with the lowest lying Kaluza-Klein mode mass $m_1 = 0.44$ and $l = 2$. 

$u_{0,2}$ and $u_{1,2}$ in Linear Scale

$u_{0,2}$ and $u_{1,2}$ in Logarithmic Scale

$1 = 2, m_1 = 0.44$

General Relativity Prediction

Prediction From Brane World

Sumanta Chakraborty

QNM and Higher Dimensions
Figure: Time evolution of the master mode function $u_{n,l}(t)$ for general relativity ($n = 0$) as well as with the lowest lying Kaluza-Klein mode mass $m_1 = 0.44$ and $l=3$. 

Sumanta Chakraborty | QNM and Higher Dimensions
We have discussed how the presence of extra dimensions will modify the black hole perturbation equations.

Possible modifications of the black hole quasi-normal modes and distinct features.

Late time behaviour of the black hole perturbations.
Thank You
At late times the frequencies can be written in an analytical manner, such that,

\[ f_n = z_n e^{27 - \left( \frac{d}{\ell} \right) (0.1 \text{mm}/\ell)} \text{Hz} \]